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NON-LINEAR INTEGRAL EQUATIONS FOR THE FINITE SIZE EFFECTS IN THE INTEGRABLE DEFORMATION OF THE QUANTUM $O(3)$ NON-LINEAR SIGMA MODEL

Elaborato finale

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"Anyway"
Juliet, at the window.

Questa tesi
è dedicata a Irene
solo io e lei
sappiamo perché

Contents

Introduction

The forefront of the present research in the most fundamental understanding of Nature is twofold: *Particle Physics* deals with the microscopic structures of the World while *Cosmology* investigates the largest known scales of the Universe.
These two apparently far disciplines are actually deeply connected and every day more evidence accumulates that at the deepest level they must be described by a theory that unifies the particle description through Quantum Fields with a consistent Quantum Theory of Gravity that includes General Relativity as a low energy limit.
Any physical theory ever proposed hasn't produced so many physical results with such a good agreement with experimental measures as the Standard Model of Particles (SM). The SM is a quantum field theory which contains the Electroweak Quantum Field Theory (EW) and the Quantum Chromo-Dynamics (QCD). It describes elementary particle interactions through a unified principle of gauge invariance and accounts for 3 of the 4 fundamental forces of Nature: the Quantum Electromagnetic interaction (QED), the Weak force, responsible of β-decays, and the QCD that keeps the quarks confined into hadrons and it is at the base of nuclear forces.
In the EW sector of the SM, quantities can be computed perturbatively, with the help of the celebrated techniques of Feynman diagrams, leading to results in astonishing agreement with the experimental measures. Moreover, very recently, the discovery at LHC of the Higgs boson has completed the general picture of the SM.
The QCD sector is, instead, more puzzling. Perturbative calculations are viable only at very high energy due to the phenomenon of asymptotic freedom, giving predictions only for the physics of hadronic jets in high energy collisions in particle accelerators. The interesting part of QCD is, instead, the physics of its bound states at low energies which is plagued by non-perturbative phenomena, like confinement. To access these phenomena theoretically, a whole set of mathematical approaches to QFT has to be developed. It goes under the name of non-perturbative QFT.

The peculiar properties of QCD, namely asymptotic freedom and confinement, are shared by a large class of two-dimensional QFTs, the so-called *Non-Linear Sigma Models* (NLSMs). They can be considered as toy models useful to investigate the main issues of QCD in a simpler setup. We shall see in the next lines that there are much deeper reasons to investigate NLSM, but this was the original motivation that convinced theorists to investigate them in the seventies.
In spite of its great successes, the SM is not a really satisfactory theory. It fails including the Quantum Theory of Gravity. In fact, Einstein Gravity is non-renormalizable at any perturbation order, making Feynman approximation method useless. Moreover, it suffers of some conceptual problems that make difficult to reconcile it with a genuine quantum physics.
It is reasonable to suppose that the impossibility to bring Gravity into SM is because of the space-time on which the model is defined is continuous. More efforts have been made in order to *quantize* the space-time by introducing some sort of non-locality at the Planck scale of $l_p \sim 10^{-33}$ cm. Many theories were born in hope to find a good Quantum Gravity theory which has the Einstein Gravity as its classical, low-energy, limit: loop quantum gravity, non-commutative space-times, asymptotically safe theory, etc...
Nevertheless, many Quantum Gravity theories suffer from a lot of conceptual and theoretical problems or from many approximations. Among all these "theories of everything", the String Theory (ST) has produced the most relevant results, albeit they are all theoretical and not testable experimentally, for the moment. It has created a quite satisfactory picture that not only quantizes gravity, but at the same moment produces a unified description of all forces and states of matter in a theory not plagued by divergences or anomalies, unlike most of QFTs. We do not want to claim that string *is* the truth, but some of its issues have helped our comprehension of the physical world, so it is worthwhile to understand better this approach.
We underline here that a propagating string, by definition a 2 dimensional mathematical object, is described in a physical way by the same Lagrangian of a two dimensional field theory that turns out to be again a NLSM. Improvement and discoveries on ST depend also on the study of NLSMs and this is what we want to do in this thesis. To be more precise, we shall study a quantum deformation of a peculiar NLSM, the $O(3)$ NLSM. The fields of the NLSM are interpreted, in the ST language, as the space-time coordinates. In this sense, studying the quantization of the NLSM, we are studying the quantization of space-time.
Quantum gravity is experimentally unaccessible on Earth, because of the high energy requested and ST is not an exception. As said, quantum aspect of Gravity becomes evident at the Planck scale, which can be quantified by

the Planck energy $E_p \sim 10^{19}$ Gev. The most powerful, and expensive, particle accelerator, the LHC, can't go beyond 14 Tev. Nevertheless, astrophysical observations could offer, also in the near future, new accessible "windows" for observations, in order to verify, exclude or constraint ST hypothesis.

So far, the best classical description of Gravity Forces has been proposed by Albert Einstein under the name of General Relativity and not even one classical experiment has called it into question. From Einstein equations, thanks to the early works of Weyl and Friedmann, it has been possible to study the evolution of the space-time as a whole physical system. The study of the space-time evolution on large scale is called Cosmology. The standard model for Cosmology is called "Lambda-Cold Dark Matter" model (ΛCDM) and, presently, it is the most appointed cosmological model of the universe. It describes the universe from its birth until nowadays, including Cold Dark Matter and Dark Energy.

At time 0 the Universe had infinite temperature and space-time and energy were enclosed in a physical singularity. According to General Relativity, this singularity is theoretically unavoidable, thanks to a theorem by Penrose and Hawking. At this initial time the Big Bang happens. From $t = 0$ and $t \sim 10^{-43}$ seconds, the Universe underwent its first stage after the Big Bang, called the *Planck era*. Probably, at this time the 4 forces were glued together in a unique fundamental Force. The next epoch, the *Grand Unification Theory* (GUT) *era*, started when the Force of Gravity separated from the other three forces (at these energies, they were interacting under one unknown Force which GUT theories try to describe). At this time, the universe expanded exponentially (inflationary epoch) for a finite time interval. After $t \sim 10^{-11}$ seconds from the Big Bang, GUT force split into two different forces, EW and Strong Force. The nucleosynthesis of nuclei heavier then Hydrogen (Helium, Lithium, Beryllium) started at $t \sim 10^2$ sec. Then, at the *recombination era*, at $t \sim 3 \times 10^5$ years, electrons recombined with nuclei and radiation was able to freely propagate, originating the Cosmic Microwave Background.

Cosmology, for those reasons, can't be a general relativity theory at all: the presence of SM particles is fundamental in the evolution of the universe. Moreover, SM can't describe the first stages of the Universe. For all these reasons, a Quantum Theory of Gravity is needed.

Furthermore, many classical systems (Black Holes, accretion on compact objects, Big-Bang singularity, etc...) undergo singularity problems. It is plausible that these singularities arise only in a classical description of gravity. Moreover, the Dark-Energy/cosmological constant issue revealed by the accelerated expansion, often related to quantum vacuum fluctuations, might get more insight, possibly, by a Quantum Theory of Gravity. All these considerations point to the fundamental issue of Quantum Gravity also from the astrophysical point of view and we believe that deepening our understanding of ST is crucial in this respect. ST is a very complex set of different and mathematically very sophisticated results produced in almost 40 years of efforts. The puzzle is far from complete and any new piece of result is welcome.

In this thesis we address the particular problem of the NLSM role in ST. As we shall briefly explain in Chapter 1, there is a direct link between the ST action and the NLSM Lagrangian. However, in recent years, a new correlation has appeared through the so-called gauge/string duality, also known as AdS/CFT correspondence.

Actually, studying the motion of a string on a curved maximally symmetric space, the NLSMs describing the 2d dynamics are not anymore conformal, but develop a mass gap. Nevertheless, they continue to be integrable.

Maldacena [1], studying a particular ST setup on a $\mathrm{AdS}_5 \otimes S^5$ space, was able to find a correspondence between this theory and a well known gauge (supersymmetric) theory, the $N = 4$ super Yang-Mills CFT living on the border of Ads_5 space, namely a $\mathrm{R}^{(3,1)}$ Minkowski space. This fact is known as AdS/CFT correspondence and it turns out to be a sort of "dictionary" between physical quantities in ST and those computed in the gauge QFT. The interesting fact is the relation between coupling constants g of these two theories, $g_{\text{gauge}} = 1/g_{\text{string}}$. This means that the weak coupling (perturbative) regime of one theory is the strong (non-perturbative) regime of the other.

This thesis has the aim to study integrable QFTs defined on a 2 dimensional space-time. By definition, a field theory is integrable if it has an infinite number of conserved charges. Starting from a field theory formulation, i.e. from a Lagrangian, it is possible to exploit the integrability of the theory. Analogously, if the theory is integrable, the S-matrix, whose elements are defined to be the scattering amplitudes between particles of the theory spectrum, turns out to have the very useful property of factorizability in products of two-particle S-matrices. These latter can be determined by imposing unitarity, crossing symmetry, analyticity and other symmetries.

Once they are known, in principle, one can evaluate every observable of integrable models: exact mass spectrum, coupling constant values, correlation functions, etc...

Of course, in some cases, it is technically difficult to find them out. The problem then becomes the one of developing new mathematical tools to attack the calculation of physical quantities.

Many 2-dimensional integrable models were deeply analyzed in this respect. In particular, if such theories are conformal, i.e. they are invariant by a local change of scale, they are integrable.

As mentioned, the integrability of these systems is a very powerful tool also for the investigation in 4 dimensional QFT or ST, thanks to Maldacena conjecture. In many cases, we deal with non-conformal models, that can be seen as perturbations of conformal ones. Often, an infinite number of conserved quantities survives the perturbation and we deal with integrable models where we can compute also non-perturbative physics.

In this thesis, we focus on the development of tools to compute the exact dependence of energy levels (or other physical quantities) from finite size of the 2d "space" on a cylinder (that mimics a closed string). In particular, we use the Thermodynamic Bethe Ansatz (TBA), a very powerful method, developed by Al. B. Zamolodchikov, which permits to evaluate physical quantities of the model, i.e. the free energy, the ground-state energy, the central charge and other fundamental elements. We study this TBA for the $O(3)$ NLSM and for a peculiar deformation of it that keeps integrability, curiously called the "sausage sigma model" (SSM) because of the form of its target space.
It turns out that TBA is a set of many coupled non-linear equations. A first step in solving it is to notice the equivalence with an integrable system of functional equations, called Y-system. This can be very complicated, it can involve many unknowns and it can be infinite. An efficient way to solve it is to find a method to "resum" many of the auxiliary unknowns into few functions, satisfying a reduced set of non-linear integral equations, called NLIEs. With this method of resummation, the resolution of the Y-system becomes feasible.

We have obtained NLIE for the Sine-Gordon (SG) model and for the SSM. In an independent way, Destri and De Vega have found, for the SG, a non-linear integral equation in substitution of the TBA system, called, after them, DDV equation. We have checked that our method gives the same result as Destri and De Vega. Later, we have found, for the first time, NLIE for the sausage model, basing our derivation on partial previous results of Ahn, Balog, Hegedüs and Dunning.

The SSM has been the main object of this thesis. Its importance is double. It is a quantum integrable model and its comprehension is useful to better understand certain SM aspects. In the limit of high energy, moreover, the SSM tends to the Witten cigar solution [2], a bi-dimensional string theory[1] called also the Witten Black Hole. In fact, this solution admits a Black Hole metric. This model can be used as a simplified system to improve our

[1]We shall see in this work that fields in ST take values on the real, or target, space-time (the space-time on which the other forces, and us, live). A string is always defined upon a (true) 2d space-time. A 2d ST means that also the target space-time is bi-dimensional.

confidence in studying the interaction between matter and Black Holes [3]. General Relativity ceases to be predictive in the presence of a singularity. To observe a naked singularity could be a challenge in order to test quantum gravity predictions based on Witten Black Hole simplification. Classically, the Cosmic Censorship Hypothesis contrasts with this possibility, but a proof of this conjecture doesn't exist. It could be wrong, but a similar hypothesis can be formulated for the quantum system. This underlines another time the difficulties arising in studying quantum gravity. Nevertheless, different kind of pioneering experiments (f.i. [4]) are trying to check if a naked singularity does or does not exist in the real 4 dimensional World.
We give a schematic presentation of this thesis:

Chapter1 The ST is briefly introduced, from its beginning to nowadays. In the last section, we underline the relation between the action of a closed string in a curved manifold and the NLSMs.

Chapter2 First, we define and describe general NLSM in n dimension target spaces. Then, we study the classical and quantum properties of a particular NLSM, the so called $O(n)$ models. These models, classically conformal, are integrable both at the classical and at the quantum level.

Chapter3 We introduce the standard S-matrix formulation of a quantum theory in 4 dimensions and S-matrix main properties: unitarity, analyticity, crossing symmetry. Later we focus on 2d integrable quantum theories and on their main properties: the factorization of the S-matrix. We present the Yang-Baxter-Zamolodchikov-Fateev equation and the algebraic representation of particle states, thanks to which general solutions of the S-matrix is found. We also outline the bootstrap approach to define bound-states. Finally, we give the S-matrix of the $O(n)$ NLSM.

Chapter4 We introduce the TBA approach to study the finite size effects of 2 dimensional integrable theories. Then, we describe how to perturb a CFT to obtain an integrable QFT in 2 dimensions. We derive the universal Y-system from TBA equations for a wide set of models. Later, after introducing some basic integrable techniques, we find the TBA equations and the Y-system for the integrable SG model. Finally, we describe the CFT which is the UV limit of the $O(3)$ NLSM.

Chapter5 It is possible to find a suitable deformation of the $O(3)$ NLSM which keeps the integrability of the model. This is the so called Sausage Sigma Model (SSM), deeply studied in this chapter. Finally, TBA equations and Y-system are found for this model.

Chapter6 We derive the NLIEs for the SG and, for the first time, for the SSM. This and the related development of a general and consistent method of resummation of TBA equations into NLIEs, constitute the main original results of this thesis.

Chapter 1

The importance of being a string

String theory (ST) is quite an old theory, born in 1968 with Veneziano amplitude, in the realm of Dual Models. At that time few theories, Quantum chromo dynamic (QCD) and Dual Models over all, seemed to go in the right direction to understand hadrons nature and their interactions. But in a series of experiments at SLAC the parton-model gave its first evidences (1974): QCD and, later, Standard Model (SM) became soon the unique language of particle physics.

Nevertheless, physicists realized that ST might have worked well not only for the strong force and hadrons, but also for all particles with high spin. Furthermore, they began to see a series of nice features -like other dimensions and supersymmetry- to come naturally from this theory. These results, at first sight impossible to endorse, turned out to be some of the best features of ST attempting to describe a unified theory of gravity and other forces. We shall take an extremely fast tour around ST world, concentrating our attention on its main achievements and on its relations with non-linear sigma models, which are the main characters of the next chapters.

We give a very fast explanation (see, for instance, [8]) of the reason for string theory, from its birth until nowadays, and in the last section we write the action of the string in curved space-time, seeing that the same action could be the description of a particular Non Linear σ Model, the $O(n)$ NLSM.

1.1 From hadrons to strings.

In the '60s, scattering experiments gave evidences of a large series of massive particles with spin J greater than 1, interacting each other through the strong force. It was immediately clear that those particles, called hadrons, were not all fundamental.

1.1.1 The wrong side of the path.

A certain number of heuristic equations had already been noticed at that time. First, the masses of those particles were related to angular momentum J by the *Regge formula*

$$m^2 = \frac{J}{\alpha'}, \tag{1.1}$$

where α', the "Regge slope", is $\sim 1(\text{Gev})^{-2}$. Second, every particle i had a particular quantum number, called *flavor* quantum number, explicated by picking a flavor matrix λ_i, which is conserved under $U(n)$ transformations. Third, scattering amplitudes of two particles into other two particles had to be proportional to Tr $(\lambda_1\lambda_2\lambda_3\lambda_4)$.

Since this amplitude is invariant under a permutation of these quantum numbers, it is also invariant under the exchange of Mandelstam variables s and t[1], where

$$s = (p_1 + p_2)^2; \qquad\qquad t = (p_2 - p_1')^2. \tag{1.2}$$

Here, p_i is the momentum for the particle arriving into the diagram and p_i' is for the particle departing from the diagram. We can figure t and s channel in Fig.(1.1).

This invariance is a key property, named *duality.*

At the tree level, in the high-energy limit, the amplitude for the t-channel scattering of a scalar field is

$$A_J^t(s,t) = -\frac{g^2(-s)^J}{t - M^2}, \tag{1.3}$$

where g is the coupling constant, s is the Mandelstam variable and M is the mass of the exchanged particle with angular momentum J. This amplitude has a bad divergent behavior for large J.

But if we take the sum over as many J-particle mediations as possible and we extend this sum to infinity, likewise considering all possible J-particle exchanges between hadrons through t-channels scattering, we obtain a polynomial in s. It is to say, an analytic function which can have single poles in s as well as in t. An infinite sum could be better than its single components, like $exp(-x)$ with $x \to \infty$, and it can converge. Furthermore, singular poles in s mean to have contributions to the amplitude also from s-channels, as

[1]The third, and last, Mandelstam variables is $u = (p_1' - p_1)^2$.

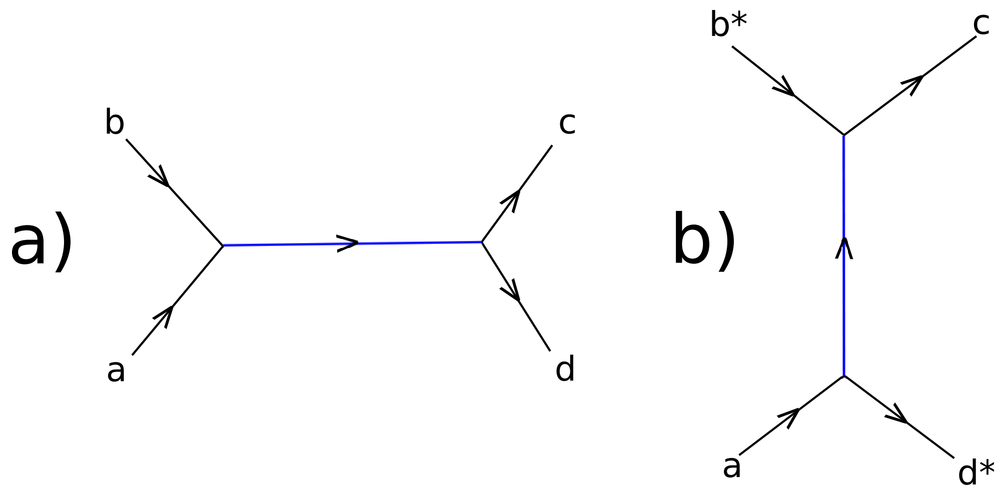


Figure 1.1: We have represented in a) the s channel for the $ab \to cd$ scatter. (b) is the correspondent t channel. Time goes from the left to the right of the figure. Arrows represent the (opposite) direction of the (anti)particle. The blue arrow represents (possible) bound states.

if we had evaluated contributions of s-channels without having considered them.
Taking the corresponding amplitude for s-channel, with M_J indicating the mass of the particle with momentum J,

$$A_J^s(s,t) = -\sum_{J=0}^{\infty} \frac{g^2(-t)^J}{s - M_J^2} \tag{1.4}$$

because of the invariance under $s \leftrightarrow t$ permutation, we find that $A_J^s = A_J^t$, i.e. it exists a *Dual property*: in the high energy limit, amplitude is a (infinite) sum over t-channel *or* s-channel and not over both, like in Quantum Field Theory (QFT).

A function having this property was suggested by Gabriele Veneziano [10]: for the Mandelstam variables t and s the *Veneziano amplitude* is

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}, \tag{1.5}$$

where Γ is the Euler gamma function $\Gamma(u) = \int_0^\infty dt\, t^{u-1} e^{-t}$. We have defined $\alpha(i) = \alpha'(i)i + \alpha(0)$.

This amplitude is consistent with a series of constraints coming from QFT, in particular: the poles are single (we are at the tree level) and the

residues are polynomial. From (1.5) we find that $M^2 = \big(J - \alpha(0)\big)/\alpha'$, which is very similar to (1.1).
In order to have (almost) all poles positive, α' must be greater than 0, space-time must be in 26 dimensions and $\alpha(0) = 1$ [11]. Here we see how others dimensions enter in ST.
As we have seen, a noticeable characteristic of the Veneziano Amplitude is the high energy behavior, which one can work out to be

$$A \sim s^{-|J|} \qquad J = \alpha(t). \tag{1.6}$$

We can switch off this amplitude whenever we want, taking $|t|$ as big as we desire! So we do not encounter divergences in the formulation of this theory. The very relevant fact here is that the convergence of (1.5) is independent from J; in the SM, Gravity is a non-renormalizable theory because bosons which mediate gravitational interactions are spin 2 massless particles, named gravitons. In general, any theory with fundamental particle with spin grater than 1 is not renormalizable. In Dual models, opposite to SM, it doesn't exist any UV divergences for any high or low spin particles.

1.1.2 Bosonic strings.

1970 saw the beginning of a series of publications (for instance, [12] and [13]) which related (1.5) to a new kind of theory based on one-dimensional objects, called strings. They could be represented in a Hilbert space, giving a physical ground to the theory. In particular, strings could be associated to particles, for instance photons and gravitons.
We start to consider the action of a massless particle moving in Minkowski D-dimensional space with a trajectory $x^\mu(\tau)$

$$S = \int d\tau \; e(\tau)^{-1} \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \qquad . \tag{1.7}$$

The $e(\tau)$ factor is needed in order to make (1.7) invariant under τ reparametrization. It is always possible to pick a gauge with $e = 1$. In this gauge, (1.7) becomes

$$S = \int d\tau \; \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \qquad . \tag{1.8}$$

Action invariance under δx^μ results in $d^2 x^\mu / d\tau^2 = 0$. Otherwise, a change from τ into τ' doesn't modify the action too, leading to the *constraint equation*

$$\frac{\delta S}{\delta e} = 0. \tag{1.9}$$

(1.9) for (1.7) is

$$\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1.10}$$

or, in other words, the lightlike geodesics in Minkowski spacetime. Finally, quantizing the system, we obtain that (1.10) is nothing but the massless Klein-Gordon equation. So, we have worked out *two* equations laid down from (1.7) invariant under x^ρ and τ variations.

Now it is time to introduce strings. A string is a mathematical curve in one dimension, open or closed, depending on the real parameter $\sigma \in [0; 2\pi]$. To study the dynamic of a string we need a sort of time parameter τ. Finally, putting the string into spacetime, we mark its position with

$$X^\mu(\sigma, \tau). \qquad \mu = 1, .., D. \tag{1.11}$$

For one value of τ, each value of X^μ for different value of σ is the position of a point of the string. Equally, it is possible to think at X^μ as values of D massless scalar fields on the 1-dimensional string.
Like a moving particle in spacetime draws a world line, a string sketches a world sheet. For this reason, if we want the equation of motion of a string from minimum action, we have to minimize the *area* of the world sheet. The first string action was found by Nambu and Goto and simplified by Polyakov introducing the new dynamical variable $h_{\alpha\beta}$ (nothing but the metric tensor of the string world sheet)

$$S = -\frac{T}{2}\int d\sigma^2 \sqrt{h}\ h^{\alpha\beta}\ \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.12}$$

Here $d\sigma^2$ is short for $d\sigma d\tau$ (generally we shall take $\sigma^\alpha = \sigma, \tau$ for $\alpha = 1, 0$ respectively). T is a constant, used to make (1.12) dimensionless. Its dimension is $[length]^{-2}$ and it will be the *tension* of the string. h is the determinant of $h_{\alpha\beta}$. This action describes the string classical motion in a Minkowski D-spacetime with D generic, but after quantization it turns to be *anomaly free* "only" in 26 dimensions.

1.1.3 Conformal invariance and constraint equations.

$h_{\alpha\beta}$ is a 2×2 symmetric matrix, but because of $\sigma^\alpha \to \sigma'^\alpha$ invariance of (1.12), the number of independent variables reduce to 1. So we can choose the *conformal gauge* $h_{\alpha\beta} = e^\phi\ \eta_{\alpha\beta}$, where the *conformal factor* e^ϕ is introduced, ϕ being a scalar.
Two words on conformal theories. We call with this name every theory invariant by a *local* change of *scale*. If a theory has a fundamental mass

or, equally, a fundamental energy (we say: *a fundamental scale of mass or energy*) it is not conformal. Conformal field theories in two dimensions are completely integrable -as we shall see in the following chapters- and we are discovering now that this simple massless ST is conformally invariant in two-dimensions.
In the conformal gauge the action can be reduced to

$$S = -\frac{T}{2}\int d\sigma^2 \eta^{\alpha\beta}\ \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.13}$$

In quantum world, X^μ transforms like a vector field in 26-dimensional spacetime and like 26 scalar fields in (1+1)-dimensions for a reparametrization of σ and τ. Nowadays, Polyakov and Nambu-Goto two-dimensional actions (and their supersymmetric generalizations) are known to be the only generally covariant actions for field theories in any number of spacetime dimensions.

Likewise for (1.7), (1.13) invariance under $\delta\sigma^\alpha$ and δX^μ leads to

$$\Big(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\Big)X^\mu = 0 \qquad \text{(equations of motion);} \tag{1.14}$$

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} = 0 \qquad \text{(constraint equations).} \tag{1.15}$$

$T_{\alpha\beta}$ is defined as the $(1+1)$-dimensional *energy-momentum tensor*.

Since X^μ are the coordinates of the string, it is possible to impose the invariance under Poincaré transformations

$$X^\mu \to \Lambda^\mu_\nu X^\nu + a^\mu. \tag{1.16}$$

But it is possible to find representations of Poincaré symmetries in the Hilbert Space, therefore particle states can bring mass and spin. In this way strings "create" particles and, since strings have infinite ways to oscillate, an infinite number of different kinds of particles could be created.
Without supersymmetry (symmetry between fermions and bosons), the closed string ground state is a tachyon, which would violate causal chronological ordering in space-time, but with supersymmetry this tachyon state is absent, thanks to a particular boson-fermion cancellation called *GSO projection* [9]. In this way supersymmetry is necessary in ST.

1.1.4 Closed strings interactions.

In this section we concentrate our attention on closed strings and their interactions. Results will be similar for open strings. We call a string *closed* if $X^\mu(0,\tau_0) = X^\mu(2\pi,\tau_0)$.

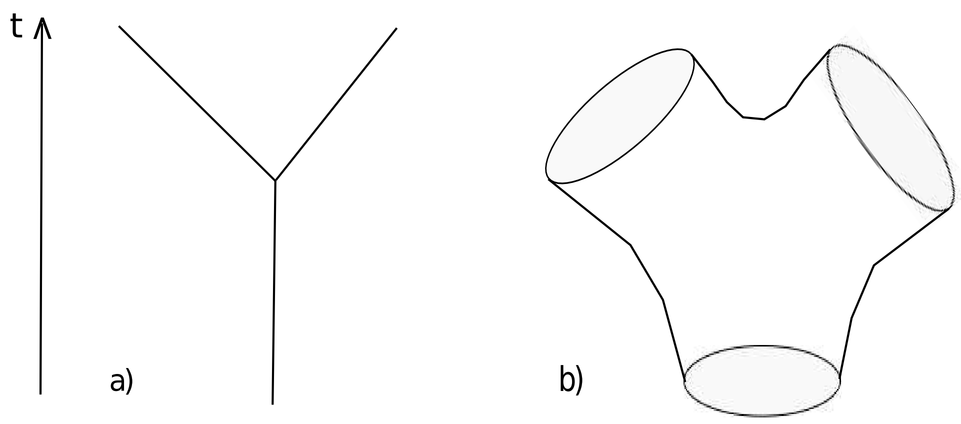


Figure 1.2: Interaction vertices in QFT and ST: in a) a point particle splits into two; in b) a closed string splits into two. In the latter case, it doesn't exist an interaction point.

Feynman diagrams describe diagrammatically fundamental particle interactions in spacetime. Any diagram can be divided in few characters: external particles -coming into and going out of the diagram- , vertex -interaction events- , internal particles -which mediate interactions- . What is a vertex? How do we image it? Like particles themselves are *point* particles also vertex are spacetime point. This means that we exactly know where interactions happen.

Things are different for strings. Instead of a world line, we have a world sheet which possibly interacts in Feynman diagram like those in Fig.(1.2). It is clear that is impossible to find a *point* (*i.e.* event) where interaction divides one string into two strings or vice-versa. This is a key point, because here we find the true difference between point particles and strings. In fact, for this reason string loop doesn't blow up amplitude addends in perturbation theory and the theory doesn't have any divergence (like Veneziano amplitude (1.7)).

It is always possible to draw a free string in diagram like that of Fig.(1.2). Mathematically, this means that choosing a particular free ST automatically determines the way for interactions. Renormalization is a mathematical trick to cut-off divergences, probably due to *point particle approach*. Strings don't need renormalization at all.

Another difference is the number of loop diagrams. For each order in perturbation expansion, we have many Feynman diagrams for point particle but only one for strings. This is because of a geometry property: two-dimensional manifolds are completely determined by handles number. One-

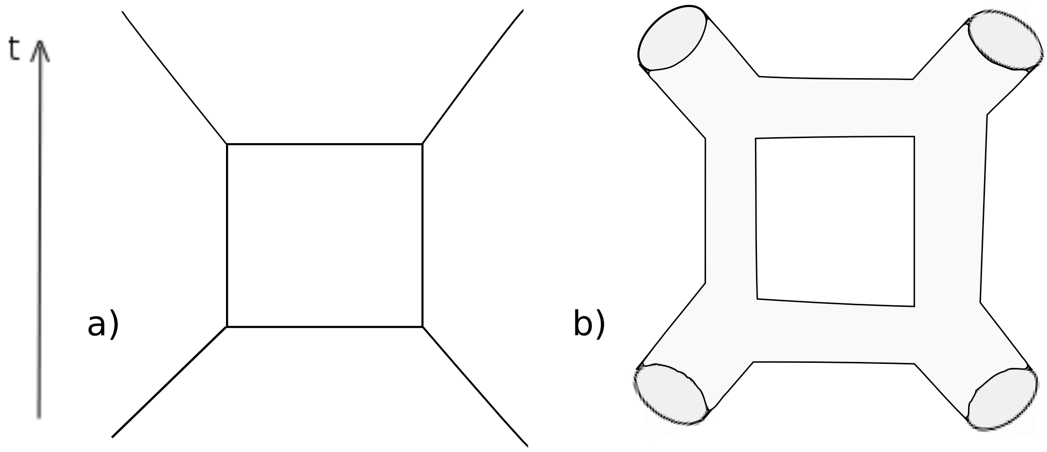


Figure 1.3: One-loop diagram vertices in QFT (a) and ST (b).

loop diagram will be the diagram with one hole, two-loop diagram will be that with two holes, and so on. In Fig.(1.3) we can see the1-loop diagram which can be represented by a torus.

1.1.5 Vertex operators.

Let's start with an example of a conformal transformation.
Take a cylinder. This is a world sheet of a free closed string propagating in the spacetime. Here, the infinitesimal relativistic invariant ds^2 is equal to $dz^2 + d\phi^2$. The change of coordinate $z = \ln r$ gives $ds^2 = r^{-2}\big(dr^2 + r^2 d\phi^2\big)$. Now, because of conformal invariance, choosing r^2 as the conformal factor, we find the metric in the plane: $ds^2 = dr^2 + r^2 d\phi^2$.
Think at the cylinder (string world sheet) in the spacetime. To $z \to -\infty$ corresponds the particle state in the infinite past, to $z \to +\infty$ the particle state in the infinite future. Transposed into the plane, these states correspond, respectively, to $r = 0$ and $r = +\infty$ (*Riemann sphere*).
If we choose another conformal factor, we find a new manifold, maybe more useful. For example, taking the conformal factor equal to $r^2\big(1 + r^2/a^2\big)^{-2}$ it's possible to obtain the metric of the sphere, namely $ds^2 = \big(dr^2 + r^2 d\phi^2\big)/\big(1 + r^2/a^2\big)$. The south pole point is the infinite past particle state, the north pole the infinite future.
Very briefly, due to conformal factor independence from any strings parameters , managing the former it is possible to project the asymptotic state of the latter wherever we want!

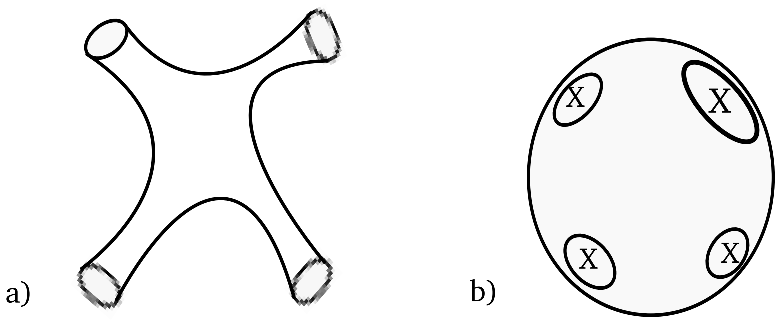


Figure 1.4: Conformal invariance makes it feasible to evaluate string diagram. The external string states in a) are projected to points, indicates as X in b).

In Fig.(1.4) we represent the already known four-string interaction. The "big ball" (topologically equivalent to a sphere) in *b*) is a diagrammatical way to resume tree diagrams. Conformal invariance permits to stretch our diagram, because distances between spacetime points lose importance and amplitude becomes pliable. Properties (quantum numbers: mass, spin, etc...) of the "stretched" external strings can't disappear. Hence, we define local scalar operators in $(1+1)$-dimensions -called Vertex operators- which 'conserve" -speaking loosely- these quantum numbers. This operators will be polynomials in X^μ and its derivatives.

It could be of some interest to show how is the amplitude in ST in path-integral formulation for M external massless, closed strings:

$$
\begin{gathered}
A(\text{M-particles}) = k^{M-2} \int DX(\sigma,\tau) Dh(\sigma,\tau) \times \\
\exp\left\{ -\frac{1}{2\pi} \int d^2\sigma \sqrt{h}\; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \right\} \cdot \prod_{i=1}^{M} V_{\Lambda_i}(k_i), \qquad (1.17)
\end{gathered}
$$

where k is the coupling constant for closed strings, k_i are string momenta and V_Λ are the vertex operators, depending on the string they are representing, each of these labelled by Λ_i.
The relevant aspect of this formula is a sort of *duality* property. The integral (1.17) is evaluated in $(1+1)$-dimensional field theory but it can be equally thought as a scattering in a field theory defined in 26-dimensional spacetime background.

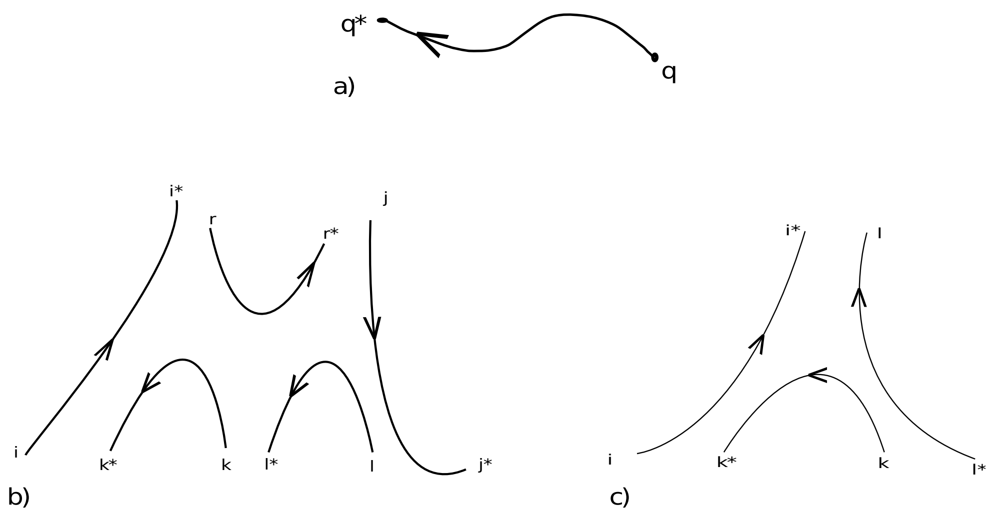


Figure 1.5: We can suppose that an open string has a "quark" and an "anti-quark" attached to both end points, as sketch in a). They can transform, respectively, in the n and the $\overline{n}$ representation of a $U(n)$ symmetry. The amplitude in b) and c) must be proportional to $\mathrm{Tr}(\lambda_1 \cdots \lambda_M)$, where M is the number of interacting strings.

1.1.6 Open strings.

We call a string *open* if $X^\mu(0,\tau_0) \neq X^\mu(2\pi,\tau_0)$. Open string ground state in 26-dimensional spacetime is a tachyon, but the first level is a massless particle of spin 1, *i.e.* a photon. Next levels are massive bosonic particles.
It is possible to have scattering between open strings, as in Fig.(1.5).

The boundaries of an open string are associated with opposite charges, like quark-antiquark. If this string breaks up, quark and antiquark match together, annihilating themselves and forming a closed string.
Imaging this configuration, good results obtained from ST attempting to describe strong force can be understood thinking about the string like a gluons' tube connecting two quarks, with a maximum tension (T) which confines the extremities.

For four-string amplitude, we recover the Veneziano amplitude

$$A = g^2 \int_0^1 dx x^{k_1 \cdot k_2}\ (1-x)^{k_2 \cdot k_3} = g^2\ B\Big(-\frac{s}{2}-2, -\frac{t}{2}-2\Big), \tag{1.18}$$

where k_i are string momenta and g is the coupling constant for open strings.
More general amplitude with both open and closed strings can be built.
Also, it is shown that the two coupling constants k and g are strongly related.

1.1.7 World-sheet Supersymmetry in ST.

So far in this chapter strings are related to bosonic particles. Because of the presence in Nature of fermionic particles, we must find the way to include them in ST. Physicists have understood that the only procedure to do that is to introduce a symmetry between bosons and fermions, that is operators which transform bosons into fermions and vice-versa. This symmetry is called *supersymmetry* (SuSy).
SuSy is necessary also to resolve another ST's shortcoming: tachyons. We have know so far that tachyons are the ground state of both closed and open strings in 26-dimension.

We want to find a generalization of (1.13). This is because we need a new vacuum (like the Higgs theory) which leads us to a new ground state. Surprisingly, it was found by Neveu, Schwarz and Ramond that adding free fermions to the theory would have solved these problems:

$$S = -\frac{1}{2\pi}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu - i\overline{\psi}^\nu \rho^\beta \partial_\beta \psi_\nu. \tag{1.19}$$

We have introduced ψ^μ, that is a Majorana spinor defined on the world sheet and depending on two variables. It can be defined as a *fermionic string.* ρ^α are the Dirac matrices in two dimensions, defined in a convenient basis

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.20}$$

(1.19) is invariant under the infinitesimal transformations

$$\begin{aligned} \delta X^\mu &= \overline{\epsilon}\psi^\mu \\ \delta\psi^\mu &= -i\rho^\beta \partial_\beta X^\mu \epsilon \end{aligned}, \tag{1.21}$$

where ϵ is a constant anticommuting spinor. These are called *SuSy transformations.* Action invariance requires that fermionic and bosonic coordinates move one into each other, becoming the "same" thing.

From this considerations, it is possible to find that the spacetime must be defined in 10 dimensions rather than in 26.

1.2 Vices or virtues?

Some ST's characteristics could seemed strange, unacceptable, unobservable: high number of dimensions, SuSy, strings, dilatons, ghosts, etc...
Of course, ST has not produced anything observable so far[2]. Nonetheless,

[2]The fact that Einstein equations come to light from ST is, for a wide set of physicists, its first observable.

physicists continue to study strings in order to find a theory which joins together the four forces of Nature. In fact, its strangeness can be seen as a powerful hint of a new, complete *theory of everything.*

1. Scattering amplitude doesn't have any divergence nor for high energy nor for high spin. Renormalization is not necessary.

2. Scattering amplitude in the low-energy limit seems to recover the behavior of Yang-Mills theory.

3. Perturbation expansion of Einstein equations is reproduced from closed strings in the low-energy limit.

4. Strings could represent both massless and massive particles of any values of spin.

5. Strings go beyond the limit of thinking particles as point-like objects. Now particles are one-dimensional curve's *vibrations.*

6. SuSy introduces a symmetry between bosons and fermions and it is necessary to include fermions in the theory. It reduces from 26 to 10 the spacetime dimension in order to make the theory consistent and it is a powerful hint in order to join the four forces.

7. 10-spacetime is another hint for the unification. In fact, from Kaluza's first attempt to unify gravity with electromagnetism [14], remembering Klein [15] and Einstein and Bergman [16], physicists began to understand that spacetime need to be *bigger* than 4-dimensions. ST needs 10 dimensions to be consistent.

8. The necessary condition to have a unitary ST is a relation between open and closed strings coupling constants, so that $k \sim g$.

Everything so far is kneaded together.

It is important to note now that constraint equations (1.15) are the string Schrödinger equations. This equation is linear and its non-linear generalization is Yang-Mills equation for the SM. Speaking about gravity, field equations are the Einstein equations, which are also non-linear generalization of Newton Laws.
A non-linear generalization of (1.15) was found also for ST, but if Einstein (and Yang-Mills) equations are preceded by basic principle *concepts* (equivalence principle, covariance, etc...) ST's equations are not yet.

1.3 Strings in curved spacetime.

The action (1.12) is the Lagrangian in flat 26-dimensional spacetime. It is possible to evaluate strings propagation in curved spacetime simply replacing the flat metric $\eta_{\mu\nu}$ with the more general $g_{\mu\nu}(X)$. (1.12) becomes

$$S = -\frac{1}{2\pi}\int d\sigma^2\sqrt{h}h^{\alpha\beta}\ g_{\mu\nu}(X)\ \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.22}$$

If we consider a flat spacetime perturbed by a gravitational wave, we write the curved metric $g_{\mu\nu}(X) = \eta_{\mu\nu} + y_{\mu\nu}\,(X)$, where $y_{\mu\nu}$ is the perturbation. In the conformal gauge, (1.12) becomes

$$S_0 = -\frac{1}{2\pi}\int d\sigma^2\eta^{\alpha\beta}\ \eta_{\mu\nu}\ \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{1.23}$$

This is the *free field action*, the same which appears into the *free field partition function* Z_0

$$Z_0 = \int DX^\mu Dh_{\alpha\beta}\ e^{-S_0}. \tag{1.24}$$

Partition function for (1.22) in the perturbation limit will be

$$Z = \int DX^\mu Dh_{\alpha\beta}\ e^{-S_0} \times \left(1 + V + \frac{1}{2}\ V^2 + \dots\right) \tag{1.25}$$

where $V = \frac{1}{2\pi}\int d^2\sigma\sqrt{h}\ h^{\alpha\beta}\ y_{\mu\nu}\big(X\big)\ \partial_\alpha X^\mu \partial_\beta X^\nu$ is the vertex operator.
In this special case, Minkowski spacetime $\eta_{\mu\nu}$ has been perturbed by a gravitational wave $y_{\mu\nu}$. This gravitational wave is nothing but a graviton, or a closed string, from the particle physics point of view. As we have already seen, this string could be represented by a vertex operator, namely the V just defined before.
Since the first part without V of (1.25) represents the free-string propagation, (1.25) describes the interaction between a *generic* string and a graviton (a mode of the closed string).

(1.22) could be written in the conformal gauge as

$$S = -\frac{1}{2\pi}\int d\sigma^2\ g_{\mu\nu}(X)\ \partial_\alpha X^\mu \partial^\alpha X^\nu. \tag{1.26}$$

This is the proper Action of a *nonlinear sigma model* (NLSM). If we want to study the behavior of a string in curved spaces, it is necessary to go deep into the study of the fascinating properties of the NLSM.

In particular, studying *quantum* NLSM, we are quantizing the fields X^μ. As they play the role of space-time coordinates in the string interpretation, we see that we are attacking the point of quantizing space-time on a curved metric $g_{\mu\nu}(X)$. In this sense, ST is a viable proposal for a Quantum Gravity framework and the quantization of NLSMs is a fundamental tool in this investigation.

Chapter 2

Nonlinear Sigma Models

Nonlinear sigma models (NLSMs) are very useful tools in many fields of Physics. They share many remarkable features like renormalizability, asymptotic freedom, asymptotic safety, solitons, confinement, spontaneous symmetry breaking, etc. They are used to study chiral quantum systems, low-energy effective physics of hadrons and QCD, solitons, condensed matter, topology, geometry, etc.
We are mostly interested in studying NLSMs because of their similarity with Einstein's theory of gravity. We have seen in the previous chapter one of these analogies. It is possible to show that the Einstein-Hilbert action can be seen as particular NLSM with a constant coupling having the same dimensions of the Newton constant. The only difference is a non-local term appearing in the Hilbert action, called *cosmological*, in the effective-field approach.
The most relevant aspect for us is the relation between string action and $O(n)$ NLSM, a particular theory which is integrable both at the classical and at the quantum levels. Integrability in string theory has become more and more important, in relation to the Anti-De Sitter/Conformal Field Theory correspondence (hereafter AdS/CFT).

NLSMs are described by a free field Lagrangian plus a constraint for the fields. In fact, we impose the field to live on a particular manifold, the coset space G/H, where G and H are the Lie group refer to, respectively, $\mathfrak{g}$ and $\mathfrak{h}$ algebras, and $H \subset G$.
Starting from a very general description of the model, we go deep into the details of 2d NLSM, taking care to illustrate many of their relevant aspects. Our main object of interest, the $O(n)$ NLSM, is an integrable theory, conformal at the classical level, about which we know the integral of motions, the spectrum (at the quantum level) and the S-matrix. The scope of this thesis will be analyses one of these model, the so-called $O(3)$ NLSM and its *quantum deformation*, called *the sausage model*.

2.1 General remarks on (euclidean) NLSM.

A (euclidean)[1] NLSM is described by the Lagrangian

$$S = \frac{1}{2\lambda^2} \int dx^n \partial_\mu \phi^\alpha \partial^\mu \phi^\beta g_{\alpha\beta}(\phi), \tag{2.1}$$

where ϕ^α is a map (or a scalar field), with $\alpha = 0, ..., d-1$ from a n-dimensional manifold X to a d-dimensional *target* manifold M, $g_{\alpha\beta}$ is the positive-definite *internal metric* depending on ϕ and λ is the coupling constant (it is the loop-counting parameter including the role of $\hbar$). We consider X as a sort of spacetime, usually defined as the *true spacetime*. In view of what was said in section 1.3, it is possible to think at X as a background space upon which we define the *real spacetime* coordinates ϕ^α (or X^μ in (1.26)). Here we shall consider X endowed with the euclidean metric $\eta_{\mu\nu} = \delta_{\mu\nu}$.

From the field point of view, we can expand the *function* $g_{\alpha\beta}$ near ϕ_0 in order to obtain an infinite series of terms which are the coupling constants of the theory

$$\frac{1}{\lambda^2}\, g_{\alpha\beta}(\phi) = \frac{1}{\lambda^2}\Big(g_{\alpha\beta}(\phi_0) + \partial_c g_{\alpha\beta}(\phi)\Big|_{\phi_0} \phi^c + \partial_d \partial_e g_{\alpha\beta}(\phi)\Big|_{\phi_0} \phi^d \phi^e + ...\Big) \tag{2.2}$$

with $\partial_a = \partial/\partial\phi^a$.
If ϕ^α belongs to a *compact* manifold, some term of (2.2) becomes linearly dependent on the others, and if we have enough *isometries*, the infinite number of coupling constants could shrink to 1. To help us get better acquainted, thinking at ϕ^α as a coordinate of M; we can study the transformation

$$\phi'^\alpha = \phi^\alpha + \epsilon \cdot \xi^\alpha \tag{2.3}$$

where $\epsilon \in \mathbb{R}$ and $|\epsilon| \ll 1$ and ξ^α is, for now, a generic scalar field defined in X. This transformation induces a change in the metric

$$g'_{\alpha\beta}(\phi') = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(\phi). \tag{2.4}$$

If the metric is *covariant* (*i.e.* form-invariant) with respect of (2.3), it respects the functional equation

$$g'_{\alpha\beta}(\phi) = g_{\alpha\beta}(\phi) \qquad \forall\phi \tag{2.5}$$

and so

$$g'_{\alpha\beta}(\phi') = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g'_{\mu\nu}(\phi). \tag{2.6}$$

[1] In the following we shall consider only this kind of NLSM.

If this equation is satisfied, (2.3) is called an *isometry.*
Substituting (2.3) into (2.6) we obtain a very restrictive condition on ξ^{α}[2]:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{2.7}$$

Every vector satisfying (2.7) is called a *Killing vector* of the metric.
This internal isometry leads to its correspondent Noether current and charge. If G[3] is a Lie group of symmetry which leads to (2.3) and $\mathfrak{g}$ is its Lie algebra, the Killing vector ξ^α follows the rules of $\mathfrak{g}$. We shall say that the theory is G-invariant. Charges introduce constraints to the theory, making possible a reduction of the number of coupling constants.

It is useful to write (2.1) in a different way in order to see better that NLSM are interacting models. Let's write

$$g_{\mu\nu}(\phi) = \delta_{\mu\nu} + h_{\mu\nu}(\phi). \tag{2.8}$$

Notice that we *do not* consider here a perturbation of the metric.
With a new normalization, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^\alpha\partial^\mu\phi_\alpha + \frac{1}{2\tilde{\lambda}^2}\partial_\mu\phi^\alpha\partial^\mu\phi^\beta h_{\alpha\beta}(\phi). \tag{2.9}$$

We have extracted the *free-field* term from (2.1) (first term in the r.h.s.), leaving alone the *interacting term* (second term in the r.h.s.). We can easily see that the equations of motion are

$$(\Box\phi^\alpha)h_{\alpha\gamma} + \partial^\mu\phi^\alpha\frac{\partial h_{\alpha\gamma}}{\partial\phi^\rho}\partial_\mu\phi^\rho + \frac{1}{2}\partial_\mu\phi^\alpha\partial^\mu\phi^\beta\frac{\partial h_{\alpha\beta}}{\partial\phi^\gamma} = 0 \tag{2.10}$$

If $h_{\alpha\beta}$ is not a function of ϕ, then (2.10) reduces to the massless Klein-Gordon equation $\Box\phi_\gamma = 0$.
It is important to notice that we are dealing with a potential. Moreover, we shall see that, in some cases (for instance when the target manifold is *homogeneous*), it is possible to *gauge* some internal symmetry (spontaneous symmetry breaking) and to create Goldstone bosons.

2.2 $O(n)$ NLSM.

As an example of NLSM theory with a single coupling constant we take the $O(n)$ NLSM, whose action is invariant under $O(n)$ transformations *on the*

[2]We use the notation of an index preceded by a semi-colon to mean covariant derivative w.r.t. that coordinate. If the index is instead preceded by a simple comma, it means an ordinary derivative

[3]Usually $M = G/H$, where H is a subgroup of G and G/H is the *coset space* of G with respect to H. For more details see the Appendix A.

field.
The $O(n)$ NLSM is defined by

$$S = \frac{1}{2} \int d^d x \partial_\mu \phi_\alpha(x) \partial^\mu \phi^\alpha(x). \tag{2.11}$$

with the condition

$$\phi_\alpha \cdot \phi^\alpha = 1. \tag{2.12}$$

We recognize in (2.11) *two* scalar products, one in X and one in M. This action is the most general $O(n)$ invariant with at most *two* derivatives, up to a multiplicative constant.

We know that X is a d-dimensional euclidean space, but what can we say about M? Let $g(x)$ be an element of $O(n)$. If we define the $n-vector$ $u = (1, 0, ..., 0)$, $\forall x \in X \; \exists \; D(g(x))$, element of the fundamental representation of $O(n)$, such that $\boldsymbol{\phi}(x) = D(g(x))u$. Since u is invariant under the action of $H \subset O(n-1)$ (H is called the *stabilizer* or the *little group* of u[4]), then u is left unchanged multiplying g by H to the right. So $\boldsymbol{\phi}$ remains invariant. We have just found that $M = O(n)/O(n-1)$ and that it exists an isomorphism between M and the sphere S_{n-1}. M is an homogeneous and symmetric space.

In order to obtain (2.1), let us define a new parametrization for $\boldsymbol{\phi}$

$$\boldsymbol{\phi}(x) = \begin{cases} \sigma(x) \\ \pi^i(x) \end{cases} \tag{2.13}$$

in which $\pi(x)$ is a $n-1$-dimensional field ($i = 0, ..., n-2$) and $\sigma(x)$ is a function of $\pi(x)$ trough equation (2.12). We can solve this equation if

$$\sigma(x) = \left(1 - \pi(x)^2\right)^{1/2} \tag{2.14}$$

We can decompose the set of generators of $O(n)$ in the set of $O(n-1)$ generators *plus* the generator of the transformation

$$\delta\pi^i(x) = w^i \left(1 - \pi(x)^2\right)^{1/2} \tag{2.15}$$

The transformation of the σ field is related to (2.15) and it is

$$\delta\sigma = -w \cdot \pi \tag{2.16}$$

The action (2.11) becomes

$$S = \frac{1}{2} \int d^d x \partial_\mu \pi^i \partial^\mu \pi^j g_{ij}(\pi) \tag{2.17}$$

[4]See Appendix A

exactly the same as in (2.1). Here g_{ij} is

$$g_{ij} = \delta_{ij} + \frac{\pi_i \pi_j}{1-\pi^2}, \tag{2.18}$$

which is the metric on the sphere S_{n-1}.
As we have written below equation (2.4), the metric (2.18) is covariant under transformations generated by the Lie Group $O(n)$[5] and (2.15) is an isometry.

2.2.1 Quantization of $O(n)$ NLSM and perturbation theory.

The generating function of our theory is

$$Z(J) = \int \left\{ d\pi(x)\big(1-\pi^2\big)^{-1/2} \exp\left[-\frac{1}{\lambda^2}\Big(S(\pi) - \int d^d x J(x)\cdot\pi(x)\Big)\right]\right\} \tag{2.19}$$

where $J(x)$ is the source function, λ^2 is the coupling constant of the theory (as explained before) and $d\pi(x)\big(1-\pi^2\big)^{-1/2}$ is the $O(n)$ invariant measure of the path integral.
It's important to notice that, in order to eliminate the infinite contribution to the action

$$\prod_x \big(1-\pi^2(x)\big)^{-1/2} \sim \exp\Big[-1/2\, \delta^d(0) \int d^d x \ln\big(1-\pi^2(x)\big)\Big] \to \infty, \tag{2.20}$$

we have to find a "good" regularization method. In (2.20), $\delta^d(0)$ is the Dirac distribution.

Minimizing (2.17), we find an infinite number of minima, that is

$$|\partial_\mu \boldsymbol{\phi}(x)| = 0 \qquad \Rightarrow \qquad \boldsymbol{\phi}(x) = \text{const.} \tag{2.21}$$

Over this infinity, we choose the simplest choice: $\pi(x) = 0$, that is $\boldsymbol{\phi}(x) = u$.
Now, if the coupling constant λ^2 is little enough (and it must be, if we want to deal with a perturbative theory), we can say that the field that contributes to the functional integral (*i.e.* the field which minimizes the argument of the exponential) is proportional to λ and, if we expand near $\pi = 0$ we find that

$$|\pi(x)| \sim \lambda \tag{2.22}$$

These are the only considerable contributions to the perturbative expansion. Terms of order 1 or greater shrink the exponential and do not contribute to

[5] The space in which ϕ^α takes values is $O(n)/O(n-1)$ and it is homogeneous *with respect* to the Lie Group $O(n)$. See Appendix A.

the functional integral. Two things are very important so far: first, the perturbation theory is not affected by the restriction of parametrization (2.14); second, we can integrate freely over $\pi(x)$ between $-\infty$ and $+\infty$ because of the restriction $|\pi(x)| \leqslant 1$.

As we have already noticed below (2.2), isometries imposed by $O(n)$ invariance make it possible the existence of only 1 coupling constant, actually λ. Furthermore, if we absorb λ in $\pi(x)$ and expand (2.19) in powers of the "new" π we find that the contribution to the n-th order by vertices is proportional only to three terms:

$$(\partial\pi \cdot \pi)^2(\pi^2)^n; \qquad (\partial\pi)^2(\pi^2)(\pi^2)^n; \qquad (\pi^2)^n. \tag{2.23}$$

The propagator of this theory is

$$\Delta_{ij}(p) = \frac{\delta_{ij}}{p^2} \tag{2.24}$$

where p is the momentum of the field π. We note that this propagator indicates that the theory is massless.

2.3 2-dimensional NLSM: renormalizability and integrability

Looking at the dimension of the terms in (2.1), we find that the integrand has dimension $[length]^{\frac{1}{d-2}}$, so, in order to have S dimensionless, $\lambda^2 = [length]^{d-2}$. But if the true spacetime X has dimension 2 nothing in (2.1) needs a dimension. The infinite coupling constants in (2.2) in 2 dimensions are all dimensionless and this fact ensures the complete renormalizability of the theory.

Furthermore, NLSMs defined upon a *symmetric* 2-dimensional manifold (so far endowed with an euclidean metric) have the extremely important property to be integrable. Any field theory needs infinite conserved charges to be integrable because of the infinite number of values of a field. 2d-NLSMs on symmetric spaces exhibit infinite conservation laws which lead to infinite conserved charges, making the theory completely integrable.

2.3.1 2d NLSM with $M = G/H$.

The definition of such models is

$$S = \frac{1}{2}\int d^2x \partial_\mu \phi_\alpha \partial^\mu \phi^\alpha, \tag{2.25}$$

where now ϕ^α take values on a *reductive* ($\Leftrightarrow$ homogeneous) manifold,that is $M = G/H$, called the *left coset* gH, $g \in G$. Here H is a compact Lie subgroup of the Lie group G and its Lie subalgebra $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{g}$ of G. It is always possible to find an inner product $\big(\centerdot,\centerdot\big)$ invariant with respect to the action of $Ad_{\mathfrak{g}}(H)$[6]. We define v and $1-v$ the *orthogonal projections* of $\mathfrak{g}$ into, respectively, $\mathfrak{h}$ and $\mathfrak{m}$, where the former is the *orthogonal complement* of $\mathfrak{h}$ with respect to $\mathfrak{g}$[7] (and M is a vector space; in general it is not a Lie Algebra). So we have that $Ad_{\mathfrak{g}}(H)\ \mathfrak{m} \subset \mathfrak{m}$ and in particular

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}. \tag{2.26}$$

The invariant inner product $\big(\centerdot,\centerdot\big)$ extends uniquely to a left G-invariant Riemannian metric on G/H. Now, on G there exists two distinguished 1-forms with values on $\mathfrak{g}$, namely the left-invariant *Maurer-Cartan (MC)* form $(g^{-1}dg)$[8] and the right-invariant MC form (dgg^{-1}). We observe a natural left-invariant connection between the left-invariant Riemannian metric and the left-invariant MC form. It is possible to describe this connection by saying that the *connection form* is the *vertical part* (the v-projection) of the left-invariant MC form on G: $A = v(g^{-1}\ dg)$. We shall focus our attention on theory where $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$.

Instead of $\phi(x)$ in (2.25), we prefer to consider the *auxiliary field* $g(x)$ which takes values on the whole G[9] and which respects the following equivalence:

$$g_2(x)\ \sim\ g_1(x)\ \Leftrightarrow\ \exists\ h(x) \in H\ :\ g_2(x) = g_1(x)h(x)^{10}. \tag{2.27}$$

The key point is that H has become the *gauge* group, while G is the *global symmetry* group.
To help us get better acquainted, we translate in terms of fields what we have tried to explain in terms of algebraic space. We take a field with values in $\mathfrak{h}$ and one field with values in $\mathfrak{m}$, respectively

$$\begin{aligned} A_\mu &= v\ \big(g^{-1}\partial_\mu g\big), \\ K_\mu &= (1-v)\ \big(g^{-1}\partial_\mu g\big). \end{aligned} \tag{2.28}$$

where A_μ is the vertical part of the left-invariant 1-form and for this reason takes values only in $\mathfrak{h}$ and K_μ is the horizontal part of the left-invariant 1-form and for this reason takes values only in $\mathfrak{m}$.

[6]With $Ad_{\mathfrak{g}}(H)$ we indicate the *adjoint representation* of H on $\mathfrak{g}$.

[7]$\mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}$.

[8]g indicates an element of $\mathfrak{g}$ and dg indicates its differential.

[9]One should not confuse between the *space* G (the space spanned with the $\mathfrak{g}$-algebra) and the *group of transformation* G. The elements of former are invariant under the action of the latter.

[10]g and h are elements of the respective groups.

It is time to introduce the *covariant* derivative, in order to deal with the field H-covariance $g(x)$. It is possible to define

$$D_\mu g = \partial_\mu g - g A_\mu. \tag{2.29}$$

Under the gauge transformation $g \to g\ h$ we can see from (2.28) that A_μ transforms as $A'_\mu = h^{-1} A_\mu h\ +\ h^{-1}\partial_\mu h$, or, in other words, it transforms as a *gauge potential*. Since elements of G must be gauge covariant, we can define $K_\mu = g^{-1} D_\mu g$, since $K_\mu \to H^{-1} K_\mu H$.
As usual in gauge theories, we define the gauge covariant tensors

$$\begin{aligned} F_{\mu\nu} &= [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \\ D_\mu K_\nu &= \partial_\mu K_\nu + [A_\mu, K_\nu], \end{aligned} \tag{2.30}$$

and we enumerate the following identities:

$$\begin{aligned} F_{\mu\nu} &= -\big(g^{-1} D_\mu D_\nu g - g^{-1} D_\nu D_\mu g\big); \\ D_\mu K_\nu &- D_\nu K_\mu + [K_\mu, K_\nu] = 0. \end{aligned} \tag{2.31}$$

Everything said before becomes useful now, in the redefinition of the action (2.25)

$$S = \frac{1}{2} \int d^2x \Big(D_\mu g, D^\mu g\Big), \tag{2.32}$$

where $\big(\cdot, \cdot\big)$ is the $Ad_{\mathfrak{g}}$-invariant inner product, namely the *trace*. We can get the equation of motion from (2.29) and (2.32). It is

$$D_\mu D^\mu g - (D_\mu g) g^{-1} D^\mu g = 0. \tag{2.33}$$

We can find from (2.33) that K_μ is a covariant-conserved current and, from the Noether theorem, that j_μ is an invariant-current generated by the left G-invariance. The latter is invariant under the action of H; the former is "only" H-covariant.

$$\begin{aligned} j_\mu &= -(D_\mu g) g^{-1}, \\ K_\mu &= g^{-1} D_\mu g, \end{aligned} \tag{2.34}$$

and their behavior with respect to, respectively, derivation and covariant derivation is:

$$\begin{aligned} \partial^\mu j_\mu &= 0, \\ D^\mu K_\mu &= 0. \end{aligned} \tag{2.35}$$

We change our coordinates from $x = (x^0, x^1)$ to (ξ, ν), also called *light-cone coordinates*[11], defined as

$$\xi = (x^0 + x^1)/2; \qquad \nu = (x^0 - x^1)/2. \tag{2.36}$$

If we derive the term $(\xi \cdot \nu) = ((x^0)^2 - (x^1)^2)/4$ with $\partial_\xi[\partial_\nu]$ we find

$$\partial_\xi = \partial_0 + \partial_1; \qquad \partial_\nu = \partial_0 - \partial_1; \qquad \Box_{01} = \frac{\partial^2}{\partial_\xi \partial_\nu}. \tag{2.37}$$

The action becomes

$$\begin{aligned} &S = \frac{1}{2}\int d^2x \left(\partial_\xi q \cdot \partial^\nu q\right) = \frac{1}{2}\int d^2x \left(D_\xi g \cdot D^\nu g\right), \\ &\big(\boldsymbol{.}, \boldsymbol{.}\big)_q = \operatorname{tr}\big(\partial_\xi q \partial_\nu q + \partial_\nu q \partial_\xi q\big), \\ &\big(\boldsymbol{.}, \boldsymbol{.}\big)_g = \operatorname{tr}\big(D_\xi g D_\nu g + D_\nu g D_\xi g\big), \end{aligned} \tag{2.38}$$

and the equation of motion can be written as

$$D_\xi D_\nu g + D_\nu D_\xi g - D_\xi g g^{-1} D_\nu g - D_\nu g g^{-1} D_\xi g = 0. \tag{2.39}$$

Equations (2.35) become

$$\begin{aligned} &\partial_\xi j_\nu + \partial_\nu j_\xi = 0 \\ &D_\xi K_\nu + D_\nu K_\xi = 0 \end{aligned} \tag{2.40}$$

where $j_{\xi[\nu]} = -D_{\xi[\nu]} g g^{-1}$.

Now we come to the point: it is possible to define a *one-parameter family* of coordinate-dependent G symmetry transformations from which it is possible to obtain an infinite series of non-local conserved currents. This symmetry is called *dual symmetry.* There is a theorem which states:

A 2d-NLSM possesses the dual symmetry if and only if the homogeneous space G/H is symmetric.

So far, if the coset space G/H is such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ or $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$, then it is homogeneous. Symmetric means that we are only in the $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ case.

In order to rule the infinite series of conserved currents, we first define the transformation mentioned above such that $\forall\ g$, solution of (2.39), $\exists\ g^{(\gamma)}$

[11] If we start from a euclidean true spacetime, this coordinates become complex numbers.

and a $n \times n$ matrix $U^{(\gamma)}$, $\gamma \in \mathbb{R}$, which respect the properties

$$U^{(\gamma)} = U^{(\gamma)}(\xi, \nu; q), \qquad UU^T = U^T U = \mathbb{I};$$
$$g^{(\gamma)} = U^{(\gamma)} g. \tag{2.41}$$

It is important to relate the solution of the equation of motion (2.33) to that related to the action (2.25) in this way:

$$q^{(\gamma)}(x) = U^{(\gamma)} q(x) \;\Leftrightarrow\; g_2^{(\gamma)}(x) = g_1^{(\gamma)}(x) h(x), \tag{2.42}$$

where $h(x)$ is an element of the group H.

Actually, until now we have restricted $U^{(\gamma)}$ to be an orthogonal transformation. General, orthogonal transformations are not good for us, but only those which are solutions of this system:

$$\begin{cases} \partial_\xi U^{(\gamma)} = \big(1 - \gamma^{-1}\big) U^{(\gamma)} j_\xi \\ \partial_\nu U^{(\gamma)} = \big(1 - \gamma\big) U^{(\gamma)} j_\nu \end{cases} \tag{2.43}$$

The integrability condition of (2.43), due to (2.40), is

$$\partial_\nu j_\xi - \partial_\xi j_\nu + 2[j_\xi, j_\nu] = 0. \tag{2.44}$$

Action (2.38) is invariant under the transformation (2.41) and so $g^{(\gamma)}$ is another solution of (2.39).
The currents transform according to

$$\begin{aligned} j_\xi \to j_\xi^{(\gamma)} &= \gamma^{-1} U^{(\gamma)} j_\xi U^{(\gamma)-1}, \\ j_\nu \to j_\nu^{(\gamma)} &= \gamma U^{(\gamma)} j_\nu U^{(\gamma)-1}, \end{aligned} \tag{2.45}$$

and it is possible to see that they respect the equation (2.40):

$$\partial_\nu j_\xi^{(\gamma)} + \partial_\xi j_\nu^{(\gamma)} = 0. \tag{2.46}$$

Expanding (2.46) around $\gamma = 1$ we find for all fixed values of (ξ, ν), the celebrated infinite series of conserved, G-covariant, H-invariant currents:

$$\begin{aligned} &\partial_\xi j_\nu + \partial_\nu j_\xi = 0; \\ &\partial_\nu \Big\{ \frac{1}{2} \Big[\int_{-\infty}^{\xi} d\xi' j_\xi(\xi', \nu), j_\xi(\xi, \nu) \Big] - j_\xi(\xi, \nu) \Big\} + \\ &+ \partial_\xi \Big\{ \frac{1}{2} \Big[\int_{-\infty}^{\xi} d\xi' j_\xi(\xi', \nu), j_\nu(\xi, \nu) \Big] \Big\} = 0; \\ &\vdots \end{aligned} \tag{2.47}$$

Note the important property of these equations (a part for the first) of being *non-local*.

2.3.2 Classical properties of 2d $O(n)$ NLSM.

The coset between $G = O(n)$ and $H = O(n-1)$ is the target space of the so-called $O(n)$ 2d-NLSMs. This space is homogeneous and the analysis made above is still true. There is a clear isomorphism between the coset space $O(n)/O(n-1)$ and the Riemann sphere S_{n-1}, as we have just seen in section 2.2.

The action is

$$\begin{aligned} &S = \frac{1}{2}\int d^2x \partial_\mu \phi_\alpha \partial^\mu \phi^\alpha; \\ &\boldsymbol{\phi}^2(x) = 1; \end{aligned} \tag{2.48}$$

here $\boldsymbol{\phi}(x) = (\phi^0(x), \phi^1(x), ..., \phi^{n-1}(x))$ and $x = (x^0, x^1)$. In this section we shall call $\boldsymbol{\phi}(x) = q(x)$, for notation convenience. For the same reason, we prefer to work in a Minkowski true spacetime[12].

The equation of motion can be derived from (2.48) with the technique of Lagrangian multipliers (or with the method used before). We find

$$\begin{aligned} &\Box q + \big(\partial_\mu q \cdot \partial^\mu q\big) q = 0; \\ &q^2 = 1 \end{aligned} \tag{2.49}$$

The invariance of the action under dilatations implies that the energy-momentum tensor $T_{\mu\nu}$ is traceless,[13]

$$\begin{aligned} &T_{\mu\nu} = q_{,\mu} q_{,\nu} - \frac{1}{2} g_{\mu\nu} q_{,\rho} q^{,\rho}; \\ &T^\mu_\mu = 0 \qquad T_{\mu\nu} = T_{\nu\mu}. \end{aligned} \tag{2.50}$$

where $g_{00} = -g_{11} = 1$ is the usual Minkovskyan metric. Therefore, $T_{\mu\nu}$ has only two independent components.

The invariance of the Lagrangian (2.48) under the action of the group $O(n)$ generates the conserved current

$$\begin{aligned} &j^\mu = q \cdot (\partial^\mu q)^T - \partial^\mu q \cdot (q)^T; \\ &\partial_\mu j^\mu = 0. \end{aligned} \tag{2.51}$$

[12] We recognize x^0 as the time and x^1 as the space

[13] This means that the theory is conformal.

With q^T we indicate the transposed vector. The equation of motion can be rewritten in light-cone coordinates as

$$\begin{aligned} &\partial^2_{\xi\nu} q + (\partial_\xi q \cdot \partial^\nu q) q = 0; \\ &q \in S_{n-1}, \end{aligned} \tag{2.52}$$

with the correspondent continuity equation

$$\begin{aligned} &\partial_\nu j_\xi + \partial_\xi j_\nu = 0; \\ &j_\rho = q \cdot (\partial_\rho q)^T - \partial_\rho q \cdot (q)^T, \qquad \rho = \nu, \xi. \end{aligned} \tag{2.53}$$

The energy-momentum conservation in light-cone coordinates can be expressed as

$$\begin{gathered} T_\xi = \frac{1}{2}(T_{00} + T_{01}), \qquad T_\nu = \frac{1}{2}(T_{00} - T_{01}), \\ T_{\xi,\nu} = 0 \qquad T_{\nu,\xi} = 0, \\ T_\xi = \frac{1}{4} q^2_{,\xi} \quad T_\nu = \frac{1}{4} q^2_{,\nu}. \end{gathered} \tag{2.54}$$

The duality symmetry is valid in this case, leading to the system

$$\begin{cases} \partial_\xi U^{(\gamma)} = \big(1 - \gamma^{-1}\big) U^{(\gamma)} j_\xi \\ \partial_\nu U^{(\gamma)} = \big(1 - \gamma\big) U^{(\gamma)} j_\nu \end{cases} \tag{2.55}$$

q satisfies (2.42) and j_ν satisfies (2.44), (2.45) and (2.46). We can expand the continuity equations in the same way of (2.47).

The interesting thing is that we can have another family of infinite *local* $O(n)$ invariant conserved currents. In fact, it is possible to deduce them by different methods. Here we shall exhibit an example for the case $n = 3$. From (2.52) we see the orthogonality between $q, \partial_\nu q$ and $\partial_\xi q$. We can argue that they are three *unit vectors* of a $\mathbb{R}^3$ basis. We define

$$\alpha = \arccos(\partial_\nu q \cdot \partial_\xi q). \tag{2.56}$$

Having identified a basis, we express $\partial^2_{\nu\nu} q$ and $\partial^2_{\xi\xi} q$ as linear combination of $q, \partial_\nu q$ and $\partial_\xi q$.

$$\begin{aligned} \partial^2_{\xi\xi} q &= -q + 2\partial_\xi \alpha (\cot\alpha) \partial_\xi q - 2\partial_\xi \alpha (\sin\alpha)^{-1} \partial_\nu q, \\ \partial^2_{\nu\nu} q &= -q + 2\partial_\nu \alpha (\cot\alpha) \partial_\nu q - 2\partial_\nu \alpha (\sin\alpha)^{-1} \partial_\xi q. \end{aligned} \tag{2.57}$$

Evaluating $\partial^2_{\xi\nu}\alpha$ in term of $q, \partial_\nu q$ and $\partial_\xi q$ and (2.57) we can easily get the celebrated sine-Gordon equation,

$$\partial^2_{\xi\nu} \alpha = -\sin\alpha. \tag{2.58}$$

This is a complete integrable Hamiltonian system. It is possible, thanks to the *Inverse Scattering Transformation (IST)*, to find an infinite series of local conserved currents. The first two continuity equations result:

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\left(\frac{1}{2}q_{,\xi}^2\right) = 0 \qquad \text{(energy-momentum conservation);}\\
&\frac{\partial}{\partial\nu}\left(\frac{1}{2\|q_{,\xi}\|}\left(\frac{\partial}{\partial\xi}\frac{q_{,\xi}}{\|q_{,\xi}\|}\right)^2\right) = \frac{\partial}{\partial\xi}\left(\frac{q_{,\xi}\cdot q_{,\nu}}{\|q_{,\xi}\|}\right).
\end{aligned} \tag{2.59}$$

2.3.3 Quantum properties of 2d $O(n)$ NLSM.

The $O(n)$ NLSM, at the quantum level, loses its conformal invariance. Like many massive quantum theories, it has a mass gap. The important thing is that it doesn't lose its integrability.
From the Lagrangian we can find the conservation equation of the energy momentum (also called "conformal" equation)

$$\begin{cases} q_{,\sigma,\tau} - uq = 0; \qquad q^2 = 0 \\ \partial_\tau\left(\frac{1}{4}q_{,\sigma}^2\right) = \frac{1}{2}q_{,\sigma}q_{,\sigma,\tau} = u\partial_\sigma\left(\frac{q^2}{4}\right), \end{cases} \tag{2.60}$$

where $u = -q_{,\sigma}q_{,\tau}$.
The main result is the demonstration of the fact that taking the *quantum anomalies* into account transforms the law (2.59) into a decent looking conservation law, analytical in terms of the fields. This phenomenon has been called *rehabilitation of conservation laws*. Anomalies destroy conformal invariance. We can introduce anomalies adding "good" terms to the right of equations in 2.60. These terms are determined by general conditions of Lorentz ($\sigma' = \lambda\sigma$, $\tau' = \lambda^{-1}\tau$) and scale ($\sigma' = \lambda\sigma$, $\tau' = \lambda\tau$)) invariance. The only possible terms are

$$\partial_\tau\left(\frac{1}{4}q_{,\sigma}^2\right) = \beta u_{,\sigma} = -\beta\partial_\sigma\left(q_{,\sigma}q_{,\tau}\right) \tag{2.61}$$

where β is the *beta function*[14].
The next conservation law (2.59) can be written as

$$\partial_\tau\left(\frac{1}{4}q_{,\sigma,\sigma}^2\right) = \frac{1}{2}q_{,\sigma,\sigma}q_{,\sigma,\sigma,\tau}; \qquad q_{,\sigma,\sigma,\tau} - u_{,\sigma}q - uq_{,\sigma} = 0. \tag{2.62}$$

[14]see chapter 5.

Using the relations

$$q^2 = 1; \quad qq_{,\sigma} = 0; \quad qq_{,\sigma,\sigma} + q^2_{,\sigma} = 0, \tag{2.63}$$

we finally obtain

$$\partial_\tau \left(\frac{1}{4} q^2_{,\sigma,\sigma} \right) - \partial_\sigma \left(\frac{1}{4} q^2_{,\sigma} u \right) = \frac{3}{4} q^2_{,\sigma} u_{,\sigma}. \tag{2.64}$$

We can add different anomalous terms: among them there are: $\partial^3_\sigma u$; $q^2_{,\sigma} u_{,\sigma}$; $q_{,\sigma,\sigma,\sigma} q_{,\sigma,\tau}$; $q^2_{,\sigma} \left(q_{,\sigma,\tau} q_{,\tau} \right)$; etc...
It is helpful to integrate (2.64) over σ since then total divergences are eliminated. The most general term with anomaly results

$$\frac{1}{4} \partial_\tau \int d\sigma q^2_{,\sigma,\sigma} = (3 + \gamma) \int d\sigma \frac{q^2_{,\sigma}}{4} u_{,\sigma} \tag{2.65}$$

where γ is a coefficient connected with the anomaly term. Now, if we take $\partial_\sigma u$ from (2.61) we obtain

$$\partial_\tau \left[\int d\sigma \left(q^2_{,\sigma,\sigma} - \frac{(3+\gamma)}{8\beta} q^4_{,\sigma} \right) \right] \equiv \partial_\tau \mathrm{I} = 0. \tag{2.66}$$

Now, from (2.66) it is possible to demonstrate that

$$\langle n | \mathrm{I} | n \rangle = \mathrm{const} \sum_i \mathrm{P}^3_{i,\sigma}. \tag{2.67}$$

In fact, from th fact that $[\mathrm{I}, S] = 0$, we can write that

$${}_O\langle b | \mathrm{I} | b \rangle_O \; {}_O\langle b | a \rangle_I \; = \; {}_O\langle b | a \rangle_I \; {}_I\langle a | \mathrm{I} | a \rangle_I, \tag{2.68}$$

where O means OUT and I means IN. Now, In the asymptotic state, with particles that do not interact, a n-plet with momentum $P_1, \cdots, P_n$ respects equation (2.67). We can write down that

$$\sum_I P^3_{,\sigma} = \sum_O P^3_{,\sigma}, \tag{2.69}$$

where sums are made on every asymptotic OUT (IN) states. Of course, in the same manner it is possible to find that (2.69) holds after the substitution $\sigma \to \tau$.

This result implies the absence of multiple production. This result allows the S-matrix to be found exactly. The natural question arises of whether the higher conservation laws exist. The answer is clearly yes,because the absence of multiple production and the crossing symmetry of the S-matrix implies that

$$\sum P^{2n+1}_\sigma = 0 \tag{2.70}$$

Chapter 3

The 2-dimensional S-matrix

The S-matrix theory is an almost self-contained theory based upon few concepts, among which: states and wave-packets, unitarity, analyticity, energy-momentum conservation, Lorentz and CPT invariance, macro-causality. This theory leads to the concept of asymptotic states, the decomposition principle, antiparticles and it is related with quantum field theory by the Feynman rules.
We can think at the S-matrix as an organized set of amplitudes of probability of scattering events between particles of the same spectrum, but also as the operator that transform the asymptotic In-state into all possible asymptotic Out-states or also as the coefficient of the commutation rule of a peculiar algebra related to particle states.
We do not treat the discussion on symmetries of the S-matrix and we only say that S-matrix is Lorentz invariant, CPT invariant and, if the theory permits other symmetries, the S-matrix is invariant with respect to them. In order to respect these invariances, the S-matrix depends on momenta, spin and other quantities related to other symmetries.
We shall see that integrable $2d$ QFTs have a factorizable S-matrix, that is, scattering between multi-particle states can be factorized in multi two-particles-states scattering. Also, integrable $2d$ S-matrix solves the Yang-Baxter equation.

We start by illustrating why S-matrix theory must be unitary and analytic. We give a definition of crossing symmetry and we see why poles of the S-matrix are related to bound states and threshold branch points to physical and not physical regions.
Then we treat the S-matrix for massive theories, and we see that this matrix respects the Yang-Baxter-Zamolodchikov-Fateev equation and that it is factorizable into two-particles scattering. Due to these properties and its symmetries, it is possible to write down the general solution, affected by a

CDD factor, i.e. an infinite number of adjustable parameters.
Finally, we point again our attention to the $O(n)$ NLSM, but this time we start from another point of view. In addition to the usual symmetries, we impose the $O(n)$ one to the S-matrix and we find the spectrum of the model, i.e. the quantum 2d $O(n)$ NLSM.

3.1 General properties of the S-matrix

We define the S-matrix as the amplitude of the event for the state Ψ_α to become the state Ψ_β[1]. We call Ψ_α^+ the *In*-state and Ψ_β^- the *Out*-state. The In-state is the asymptotic state for $t \to -\infty$ and the Out-state is the same for $t \to \infty$. In other words

$$S_{\beta\alpha} = \left(\Psi_\beta^-, \Psi_\alpha^+\right), \tag{3.1}$$

where $(\cdot, \cdot)$ is the inner product in the Hilbert space. S is a complex value for each couple of states. If we have many events, linked together, we deal with an array of $\mathbb{C}$-values. The relevant thing is that the *rate for a reaction* $\alpha \to \beta$ is proportional to $\left|S_{\beta\alpha} - \delta(\alpha - \beta)\right|^2$.

3.1.1 Unitarity.

The In- and Out-states live in two isomorphic Hilbert spaces: therefore, the In-state can be expanded as a sum of Out-states,

$$\Psi_\alpha^+ = \sum_\beta S_{\beta\alpha} \Psi_\beta^-. \tag{3.2}$$

Every In-state may become in the far future any of the Out-states. we shall see that Out-states are *accessible* states for the In-state in question.
All these Out-state form a *complete* set of orthonormal states. If we start with a complete set of orthonormal states α and we finish with a complete set of orthonormal states β, the S-matrix must be *unitary*. In fact

$$\begin{gathered}\int d\beta S^*_{\beta\gamma} S_{\beta\alpha} = \int d\beta \left(\Psi_\gamma^+, \Psi_\beta^-\right)\left(\Psi_\beta^-, \Psi_\alpha^+\right) = \left(\Psi_\gamma^+, \Psi_\alpha^+\right) = \delta(\gamma - \alpha) \\ \text{that means} \\ S^\dagger S = \mathbb{I}.\end{gathered} \tag{3.3}$$

[1] α and β summarize all the quantum numbers of the state.

3.1.2 Analyticity.

Another important property is *analyticity*. We shall start assuming analyticity as a postulate. Almost every theory in physics has involved analytic functions and experiments do not give at least any reason to believe the opposite. Analyticity of the S-matrix gives us the opportunity to find a theory where no singularities are arbitrary, but they are all related to general principles.
In fact, it is possible to recognize that S-matrix poles are related with particles. There are many poles, perhaps infinite in number. The most important type occurs in a *channel invariant*. We define a *channel* a set of more than one particle involved in a physical process. The square of the total energy in the barycentric system of the particles is the channel invariant. The invariant for the channel c (in a covariant form) is

$$s_c = \Big(\sum_{i \in c} p_i \Big)^2 . \tag{3.4}$$

In the channel invariant region ($s_c > 0$) many poles could appear. Every pole represents a new particle, that is a bound-state, provided that it is near or in the physical, channel invariant, region ($s_c > 0,\ 0 < p_i^2 < m_i c^2$). The value of the pole in this region must be interpreted as the square of the mass of the particle. We can distinguish between stable particles, which have real masses, and ordinary resonances, which have a small negative imaginary part in their mass equal to half width, i.e. half the inverse of their mean lifetime. For instance, if we have a four-particle S-matrix[2], two ingoing and two outgoing, we shall find three non-independent channel invariants, called *Mandelstam variables*, denoted by three letters: $t,\ s,\ u$. We have already spoken about these variables in chapter 1 (See Fig.(1.1)).

A particle pole will occur not just in one channel invariant but in the invariants of all channels that "communicate" with the pole particle, i.e. if the same bound state could be obtained from different states. Two channels could *communicate* if the energy conservation permits. Different particle sets producing the same bound state are called *communicating channels*. Finally, we define the *connected part* of a particle state the particle subset which does not interact with other particles of the state. Loosely speaking, the S-matrix could be decomposed in connected S-matrices, each of them related to events occurring in a particular point in spacetime, very far from the others. In other words

$$S_{\beta\alpha} = \sum_C (\pm) S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2} \cdots \tag{3.5}$$

[2]We mean that we have four external asymptotic particles.

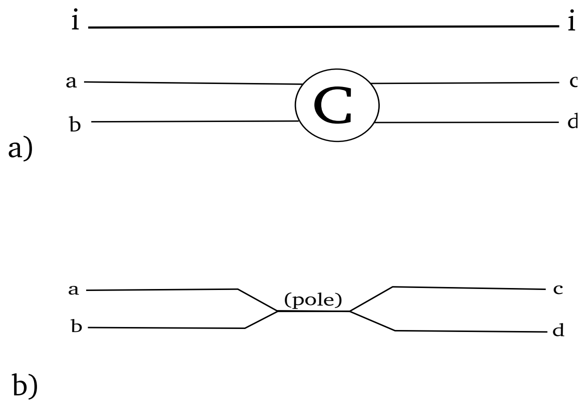


Figure 3.1: In a) a C-part between four particles (a, b, c, d) is indicated, with respect to a free particle i. In b) we see that a pole can be formed in the same connection.

C indicates the connected part in which the S-matrix is divided: the subscripts indicate, in the same C-part, all possible communicating channels; the sign in front of the sum is related to the number of fermions involved in the transition. A C-part is indicated in Fig.(3.1).

3.1.3 Crossing symmetry.

The third important property is called *crossing symmetry* and it involves *antiparticles*. The fundamental requirement is that the S-matrix be capable of describing two or more successive collisions with a macroscopic spacetime separation. For instance, if we have two reactions

$$a + b \;\rightarrow\; H + d \qquad \text{and} \qquad H + e \;\rightarrow\; f + g \tag{3.6}$$

than the reaction

$$a + b + e \;\rightarrow\; d + f + g \tag{3.7}$$

can have an S-matrix, with a pole in $s = (p_a + p_b)^2$ near the square of the H mass

$$p^2 = m_H^2. \tag{3.8}$$

In this sense, the residue of the pole is said to be *factorizable*. If the residue isn't factorizable, the particle can't exist.
If we call A the first state (in the example $a + b + e$) and B the second state, in the physical region $p^2 > m_H^2$, A is the In-state, B the Out-state. It is possible to make an analytic continuation of the whole process from the

physical region mentioned above to the physical region $p^2 < -m_H^2$, passing through unphysical regions. In this second case A is the Out-state and B is the In-state and all quantum numbers are the opposite and H becomes $\overline{H}$, that is the antiparticle of H. In other words:

$$\begin{array}{ccccc} A & \to & B & \text{through} & H \\ & analytic & continuation & & \\ \overline{B} & \to & \overline{A} & \text{through} & \overline{H} \end{array} \tag{3.9}$$

Since the factorizability of the residue must persist throughout the analytic continuation from positive to negative p, A is the analytic continuation of $\overline{A}$ and the same is for B and $\overline{B}$.
The *crossing symmetry* principle can be stated:

A single analytic function, evaluated in positive or negative timelike regions of its energy-momentum variables, represents the C-part for all reactions that differ by replacing incoming particles with outgoing antiparticles.

Why analytic continuation is so important? Because if we find the way to link all physical regions we can use a single S-matrix to describe the whole theory. We shall see in the next section that it is possible to go from one region to another passing through unphysical region also if we "meet" branch points.

3.1.4 Threshold branch points.

In addition to particle poles a second type of fixed singularities appear in channel invariants. These are the *branch point* associated with *channel threshold*. We define a channel threshold in this way

$$s_c^t = \Big(\sum_{i\in c} m_i\Big)^2. \tag{3.10}$$

The meaning for threshold derives from the beginning of the physical region *beyond* s_c^t:

$$\begin{array}{ll} s_c > s_c^t & \text{the channel is open;} \\ s_c < s_c^t & \text{the channel is closed;} \end{array} \tag{3.11}$$

where with open (closed) we mean possible (impossible).
Now we take a C-part of the S-matrix, called $A_{c'c}(s_c)$, depending on the channel invariant s_c and connecting the In-state c with the Out-state c'. Of

Figure 3.2: Here we have indicated nodes (branch points) with a label i, that is referred to the channel $s^t_{c_i}$.

course, for the energy-momentum conservation $s_c = s_{c'}$. For each c' we have different thresholds and the physical region is the real s_c region above both s_c and $s_{c'}$. It is bounded on the lower end by whichever of these thresholds is the larger. Thus, there exist thresholds which live in the unphysical region. Roughly speaking, the necessity for channel threshold branch points in physical regions arises from the fact that the dimensionality of the *physical* S-matrix changes each time a new channel opens. Obviously something sudden happens at such points, and if we are dealing with analytic functions the sudden change is manifested by a point of singularity, which we can see in Fig.(3.2).

The difference nature of the singularities mentioned above are obviously related to the different nature of their relation to the scattering process. If we want to go from a sector to another sector, we can use the *$i\epsilon$ prescription*, adding an infinitesimal imaginary part to s_c in order to go beyond the threshold and to pass over it through unphysical regions. But not every unphysical region is good. We call the only possible unphysical region where to continue analytically the *physical sheet*, a region of the whole complex Riemann surface inside clear cuts. This sheet is prescribed by drawing the cut from each normal threshold branch point in a channel invariant along the positive real axis in that variable to $+\infty$. The statement is complicated by the mass-shell and energy-momentum conservation constraints, but it has turned out, in all cases analyzed, to have a well-defined meaning when translated into an independent set of channel invariants, such as the (s, t, u) Mandelstam variables for four-particles scattering (Fig.(3.3)). Everything so far has been tested in laboratory.

3.2 The 2d S-matrix with massive particles.

We consider M ingoing particles and N outgoing particles with different masses and quantum numbers. The S-matrix for this process is[3]

$$S^{B_1 B_2 \cdots}_{A_1 A_2 \cdots} = \big(B_1(p'_1)B_2(p'_2)\cdots, A_1(p_1)A_2(p_2)\cdots\big), \tag{3.12}$$

[3]The convention is: upper indices correspond to outgoing particles, lower indices correspond to ingoing particles.

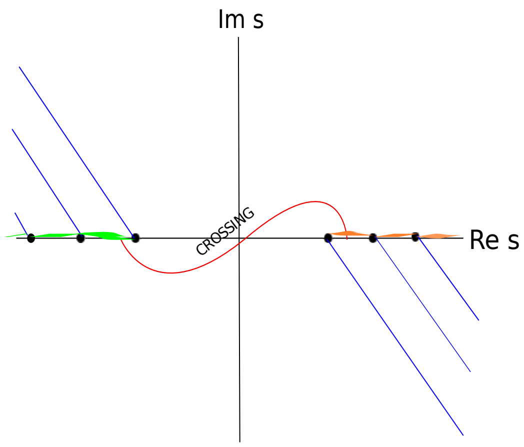


Figure 3.3: Blue straight lines indicate forbidden regions. Orange and green region are physical permitted region. Nodes indicates branch points. By crossing we are able to pass from a physical region to another without passing through any forbidden un-physical region. Thus we can have one S-matrix valid for the whole theory.

where now the top indices $+$ and $-$ are substituted respectively by the letters A and B.
In the Appendix B we present a theorem which shows that the in a $1+1$ dimensions theory with two conserved currents different from Lorentz currents, the S-matrix is factorizable and the set of final momenta is equal to the set of initial momenta. By *factorizable* we mean that the process can be factorized in processes involving only two particles. From the conservation of the set of initial momenta, these two particles either exchange their momenta or not. It depends from the mass of the particles: if particles have the same mass (we shall say: they belong to the same *mass multiplet*) they can exchange their internal quantum numbers, if the masses are different, particles can't exchange anything and the two-body scattering is purely elastic.

To help us get better acquainted, we summarize the peculiarity of the 2d S-matrix in three *selection rules*:

1. The number of particles with mass m_i is conserved and no particle is produced. Anyway, the S-matrix is non-trivial because internal quantum numbers can be exchanged between different particles in the same multiplet.
   We sketch this scatter by $A_{a_1}(p_1)+A_{a_2}(p_2)+\cdots \to A_{b_1}(p'_1)+A_{b_2}(p'_2)+\cdots$,[4]

[4] It is worth noticing that we can have different out-going particles with respect to

where A_{a_1} means *a particle of type* a_1.

2. The set of momenta is conserved, so $\{p_1, p_2, \cdots\} = \{p'_1, p'_2, \cdots\}$.

3. The S-matrix factorizes in as many as necessary two particle processes.

3.2.1 The YBZF equation and the algebraic representation of particles states.

Let us now introduce light-cone coordinates, in order to deal with rapidity θ instead of momenta:

$$p_a^0 = m_a \cosh\theta_a, \qquad p_a^1 = m_a \sinh\theta_a,$$

$$p_a = p_a^0 + p_a^1, \qquad \overline{p_a} = p_a^0 - p_a^1,$$

$$p_a\overline{p_a} = m_a^2 \qquad \text{(mass-shell condition)}.$$

Now we define, for each particle with mass m_a the quantity $a = p_a/m_a$. We find that $\theta_a = \log a$. Recall that a was a positive real number for the forward component of the mass shell; this corresponds to θ ranging over the entire real axis. The backwards component of the mass shell, found by negating a, can be parametrised by this same rapidity so long as it is shifted onto the line $\mathrm{Im}\theta = \pi$. This will be relevant when discussing the crossing of amplitudes.

We can write the first selection rule above using rapidity instead of momenta: $A_{a_1}(\theta_1) + A_{a_2}(\theta_2) + \cdots \;\rightarrow\; A_{b_1}(\theta'_1) + A_{b_2}(\theta'_2) + \cdots$. For instance, in Fig.(3.4) we see a four-particles scattering factorized in the right way. To each line corresponds a particle, to each vertex corresponds a two-particle scattering, to each slope corresponds a different rapidity (the greater is the slope the greater is the rapidity). To each vertex we have a factor $S_{I_1 I_2}^{O_1 O_2}(\theta_{12})$, where θ_{12} is the difference $\theta_1 - \theta_2$. If we have M ingoing particles, the number of collisions will be $\frac{1}{2}M(M-1)$. The S-matrix for the diagram in Fig.(3.4) is

$$S_{I_1 I_2 I_3 I_4}^{O_1 O_2 O_3 O_4}(\theta_1, \theta_2, \theta_3, \theta_4) = \sum_{j_1 j_2 j_3 j_4 k_1 k_2 k_3 k_4} S_{I_1 I_2}^{j_1 j_2}(\theta_{12}) S_{j_1 I_3}^{k_1 j_3}(\theta_{13}) S_{j_2 j_3}^{k_2 k_3}(\theta_{23}) S_{k_1 I_4}^{O_1 j_4}(\theta_{14}) S_{k_2 j_4}^{O_2 k_4}(\theta_{24}) S_{k_3 k_4}^{O_3 O_4}(\theta_{34}), \tag{3.13}$$

ingoing particles. For instance, we could have two ingoing particles with the same mass but different spin and different quantum numbers that become a different multiplet, exchanging their internal quantum numbers.

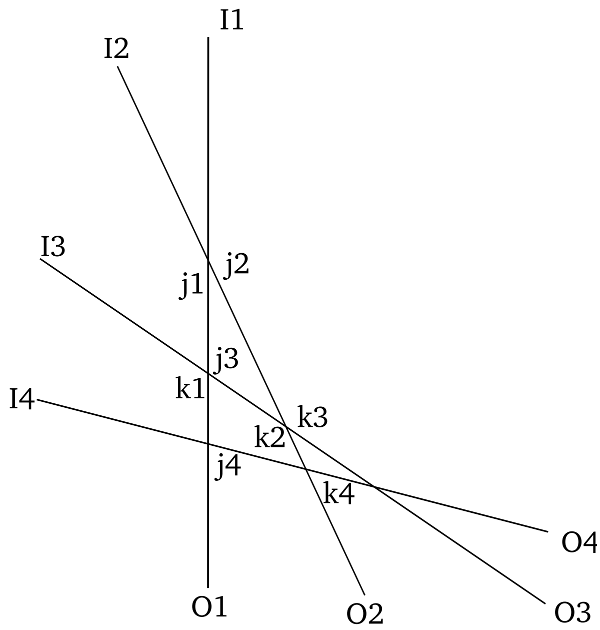


Figure 3.4: Factorized four-particles scattering. For each line we write the respective particle label.

where the small letters indicate the intermediate states and we sum over all possible intermediate states permitted by the energy and the nature of the ingoing particles. But here we may have a problem, because for the same values of rapidity we can have different diagrams, provided that we shift lines with respect to each others. Fortunately, it turns out that this diagrams are equivalent. This is possible only if the S-matrix satisfies the Yang-Baxter-Zamolodchikov-Faddeev equation, usually called *factorization equation*:

$$S_{i_1 i_2}^{j_1 j_2}(\theta_{12}) S_{j_1 i_3}^{k_1 j_3}(\theta_{13}) S_{j_2 j_3}^{k_2 k_3}(\theta_{23}) = S_{j_1 j_2}^{k_1 k_2}(\theta_{12}) S_{i_1 j_3}^{j_1 k_3}(\theta_{13}) S_{i_2 i_3}^{j_2 j_3}(\theta_{23}). \tag{3.14}$$

It is important to notice the "conservation" of the rapidity dependence. This equation can be represented in Fig.(3.5).

There is a different approach to the study of the S-matrix and it is based upon an algebraic representations of asymptotic particles states. An In-state $|A_{a_1}(\theta_1) A_{a_2}(\theta_2) \cdots\ A_{a_L}(\theta_L)\rangle$ is characterized by there being no further interactions as $t \to -\infty$. This means that the fastest particle must be on the left, the slowest on the right, with all of the others ordered in between. In fact, in order to meet in the same point, in the past the faster particles had to be farer than other particles, that is $x_1 < x_2 < \cdots < x_L$.

In a massive theory all interactions are short-ranged and so the state behaves like a collection of free particles except at times when two or more

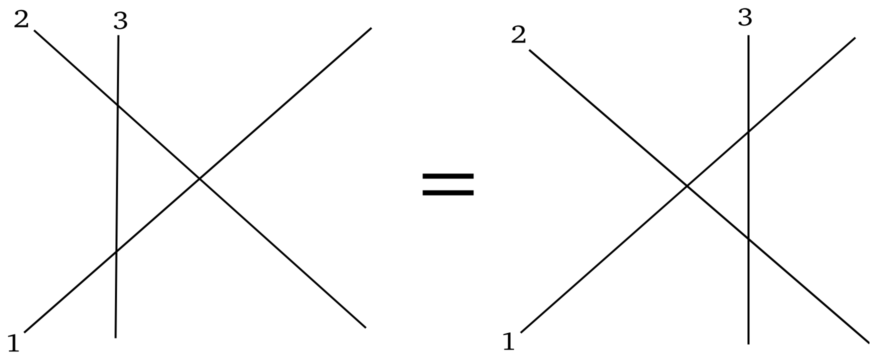


Figure 3.5: Graphic representation for the YBZF equation.

wave-packets overlap. All of this is made more precise in Appendix B.
An element of our algebra will be any kind of product of operators $\prod_{i=1}^{L} A_{a_i}(\theta_i)$, with $L \in \mathbb{N}$ and $\theta_1 > \theta_2 > \cdots > \theta_L$, acting on the vacuum $|0\rangle$ to create IN particles. The ordering of the product reflects the spatial ordering of the ingoing particles. To the operators $A_{a_i}(\theta_i)$ the name of *Zamolodchikov-Faddeev* (ZF) operators is given.
Similarly, for the Out-states we have the representation $A_{b_1}(\theta'_1)\cdots A_{b_L}(\theta'_L)$ with $\theta'_1 < \theta'_2 < \cdots < \theta'_L$, acting on the vacuum $\langle 0|$. But from the third selection rule, this relation must be written $\theta_1 < \theta_2 < \cdots < \theta_L$, with the clarification that now the time is running to ∞[5].

Now we assume the following *commutation rules*

$$\begin{aligned} A_i(\theta_1)A_j(\theta_2) &= S_{ij}^{kl}(\theta_1,\theta_2)A_l(\theta_2)A_k(\theta_1), \\ A_k(\theta_2)A_l(\theta_1) &= S_{kl}^{mn}(\theta_1,\theta_2)A_n(\theta_1)A_m(\theta_2), \end{aligned} \tag{3.15}$$

from which we find the unitarity condition and, imposing the associativity of the algebra, the YBZF equation[6], written in (3.14).

3.2.2 The two-particles S-matrix.

From the factorizability property, if we know the exact two-particle S-matrix we can know everything about every scattering events between N particles. We have just seen how to translate states into the ZF algebra. We notice that, because of the invariance of the S-matrix with respect to Lorentz boost, which shifts the rapidity by a constant, the dependence in (3.15) becomes

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_{12})A_l(\theta_2)A_k(\theta_1). \tag{3.16}$$

[5]We could write the product $A_{b_L}(\theta'_L)A_{b_{L-1}}(\theta'_{L-1})\cdots A_{b_1}(\theta'_1)$ with $\theta'_1 < \theta'_2 < \cdots < \theta'_L$.
[6]Obviously if we interpreted S_{ij}^{kl} as an element of the S-matrix.

where now the Einstein convention is switched on (considering the trivial possibility).

The invariance of the S-matrix with respect to P,T and C can be sketched, respectively:

$$\begin{aligned} S_{ij}^{kl}(\theta) &= S_{ji}^{lk}(\theta), &\qquad \text{parity } P; \\ S_{ij}^{kl}(\theta) &= S_{lk}^{ji}(\theta), &\qquad \text{time reversal } T; \\ S_{ij}^{kl}(\theta) &= S_{\overline{ij}}^{\overline{kl}}(\theta), &\qquad \text{charge conjugation } C. \end{aligned} \tag{3.17}$$

In order to discuss the analytic properties of the two-particle S-matrix, we introduce the Mandelstam variables:

$$\begin{gathered} s = (p_i + p_j)^2; \quad t = (p_i - p_k)^2; \quad u = (p_i - p_l)^2; \\ s \ + \ t \ + \ u \ = \sum_\lambda m_\lambda^2. \end{gathered} \tag{3.18}$$

In (1+1) dimensions, only one variable is independent. In fact, from selection rules, we can have $p_i = p_k$ or $p_i = p_l$. If we choose the latter, than $u = 0$ and t is fixed by the above relation, leaving the variable s as the only independent one. We can easily see that

$$s = m_i^2 + m_j^2 + 2m_1 m_j \cosh\theta_{12}, \tag{3.19}$$

with $\theta_{12} \in \mathbb{R}$. Physical values are permitted only if $s \geq (m_i + m_j)^2$ which is our first threshold branch point. The physical sheet will be represented by elements in the complex s-plane just above the Real axis after the branch point, namely $s^+ = s + i\epsilon$, where s respects the condition written above and $\epsilon > 0$ is infinitesimal.

We have just seen (section 3.1.4) where to cut the complex plane from the branch point in order to make a possible analytic continuation. Another important hypothesis is $S_{ij}^{kl}(s^*) = \left(S_{kl}^{ij}(s)\right)^*$ which, with the T invariance, becomes $S_{ij}^{kl}(s^*) = \left(S_{ij}^{kl}(s)\right)^*$, that is the condition for *real-analyticity*. Note that S is real if $s \in [(m_i - m_j)^2, (m_i + m_j)^2]$.

We can find that $(m_i + m_j)^2$ is a square root branch point. In fact, from the unitarity condition and the real-analyticity, we find that

$$S_{ij}^{kl}(s^+) S_{kl}^{nm}(s^-) = \delta_i^n \delta_j^m, \tag{3.20}$$

where $s^- = s - i\epsilon$. Call γ the curve that encircles counterclockwise the branch point. Call S_γ the analytic continuation of S along this path. Then unitarity amounts to the requirement that $S(s^+)S_\gamma(s^+) = \mathbb{I}$ for all physical values of s^+. When written in this way, the relation can be analytically continued to all s, so $S_\gamma(s) = S^{-1}(s)$. In particular, if s^+ is a point just below the cut, then $S_\gamma(s^-) = S^{-1}(s^-) = S(s^+)$, the last equality follows from a second application of unitarity. Now $S_\gamma(s^-)$ is just the analytic continuation of $S(s^+)$ twice around $(m_i + m_j)^2$. Therefore, twice round the branch point gets us back to where we started, and the singularity is indeed a square root.

In order to find another threshold branch point, we apply the crossing symmetry principle and change the In-state j and the Out-state l with, respectively, the In-state $\bar{l}$ and the Out-state $\bar{j}$. We have a new S-matrix and a new channel invariant,

$$S_{i\bar{l}}^{k\bar{j}}(\theta); \qquad t = (p_i - p_l)^2. \tag{3.21}$$

It is easy to relate t with s: $t = 2m_i^2 + 2m_j^2 - s$. The new amplitude can be obtained from analytical continuation of the "old" amplitude in the region where t permits to the system to be physical, that is $t \in \mathbb{R}$ and $t \geq (m_i - m_j)^2$. We can easily check that the t physical sheet coincides with the complex s elements just under the Real axis before $(m_i - m_j)^2$, namely s^-.
Crossing symmetry can be summarized in this equation

$$S_{ij}^{kl}(s^+) = S_{i\bar{l}}^{k\bar{j}}(2m_i^2 + 2m_j^2 - s^+). \tag{3.22}$$

Everything becomes easier if we pass from s to θ, the rapidity we have just introduced above. It can be found that

$$\theta = \cosh^{-1}\left(\frac{s - m_i^2 - m_j^2}{2m_i m_j}\right) =$$
$$= \log\left\{\frac{1}{2m_i m_j}\left[s - m_i^2 - m_j^2 + \sqrt{\left(s - (m_i - m_j)^2\right)\left(s - (m_i + m_j)^2\right)}\right]\right\} \tag{3.23}$$

and it maps the physical sheet into the region

$$0 < \mathrm{Im}\theta < \pi. \tag{3.24}$$

of the θ plane called the *physical strip*.

The previous relations can now be translated to give a list of constraints on $S(\theta)$ (we call them *primary constraints*):

1. Real analyticity: $S(\theta) \in \mathbb{R}$ only for θ purely imaginary;
2. Unitarity: $S_{ij}^{kl}(\theta) S_{kl}^{mn}(-\theta) = \delta_i^m \delta_j^n$;
3. Crossing: $S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta)$.

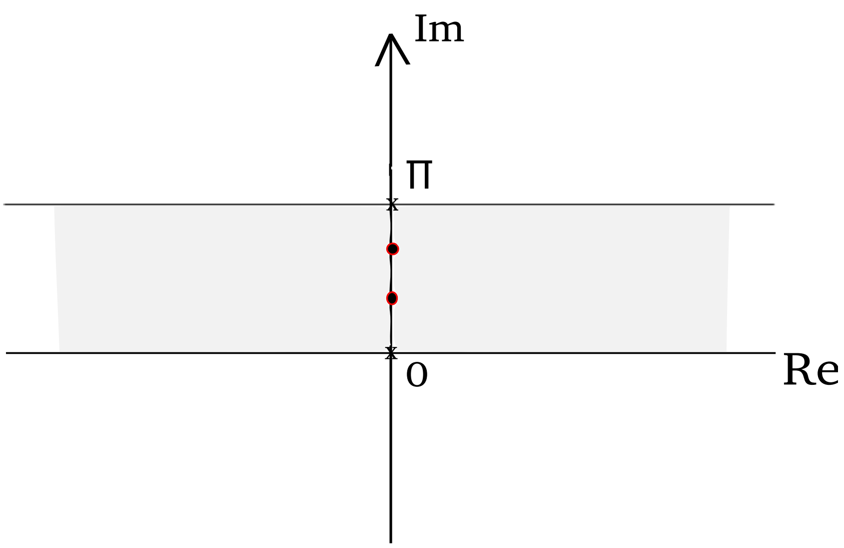


Figure 3.6: Physical strip in $\theta - \mathbb{C}$ plane. In $\theta = 0, i\pi$ we have signed branch points with a crux. Full circles indicate possible poles.

Because no particle production is permitted, if we have singular poles they must stay between $(m_i - m_j)^2$ and $(m_i + m_j)^2$ or between 0 and $i\pi$ in Fig.(3.6). Naturally, the S-matrix analytic continuation involves the whole complex plane, including poles. As explained in section 3.1.2, poles are singular and, in their neighborhood, the S-matrix can be represented as

$$S_{ij}^{kl}(\theta) \sim \frac{iF_{ij}^{kl,n}}{\theta - iu_{ij}^n}, \tag{3.25}$$

where $\theta = iu_{ij}^n$ is a generic pole of the S-matrix and $F_{ij}^{kl,n} = f_{ij}^n f^{kl,n}$, with no summation over n, is the residue of S_{ij}^{kl} at u_{ij}^n. We call the f_{ij}^n tensors the *coupling constants* of the theory. They can be diagrammatically represented as in Fig.(3.7).

3.2.3 Solutions.

We have noted earlier at the beginning of section 3.2 (and, better, in App. B) that particles can exchange internal quantum numbers only if they belong to the same multiplet. But this means that particle rapidity can change only if we have a scatter between two particles of the same mass. In this case the S-matrix is block-diagonal. Therefore, we can speak about blocks and elements *inside* the blocks. The formers are limited only by unitarity and

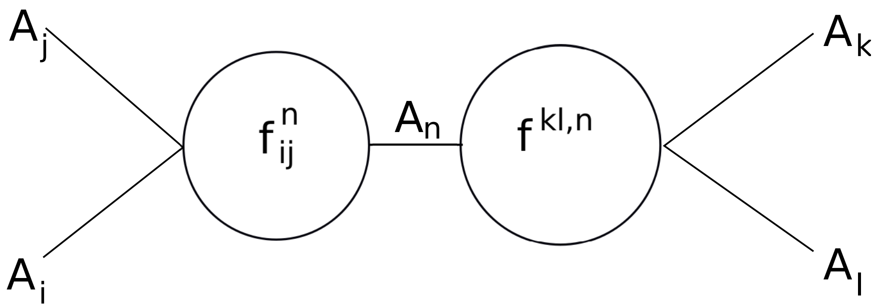


Figure 3.7: Bound states.

crossing, because (3.14) is identically satisfied:

$$S_{(i)}^{(k)}(\theta) = \delta_i^k S_{(i)}^{(k)}, \tag{3.26}$$

where block-indices are indicated in parenthesis. Inside the block, factorization is valid and elements are limited by all primary constraints.

The general solution of (3.14) can be put in the form

$$S_{ij}^{kl}(\theta) = \frac{1}{f(\theta)} R_{ij}^{kl}(\theta), \tag{3.27}$$

where $R_{ij}^{kl}(\theta)$ is a matrix which contains only *entire* functions[7] of θ and $f(\theta)$ is a meromorphic function.
$R_{ij}^{kl}(\theta)$ depends on $\lambda\theta$, where λ is a real parameter which is free to be chosen. In all known cases

$$R_{ij}^{kl}(\theta) = R_{\bar{i}l}^{k\bar{j}}(i\pi - \theta), \tag{3.28}$$

and so the crossing property becomes

$$f(i\pi - \theta) = f(\theta). \tag{3.29}$$

In the point of zero rapidity[8] we can prove that

$$R_{ij}^{kl}(0) = \delta_i^k \delta_j^l R_0, \tag{3.30}$$

and from (3.30) and (3.14), using $\theta_{23} = -\theta_{12}$ we obtain

$$R_{ij}^{mn}(\theta) R_{mn}^{kl}(-\theta) = \delta_i^k \delta_j^l Q(\theta), \tag{3.31}$$

[7] In all known results, these functions are rational, hyperbolic or elliptic meromorphic functions of θ.

[8] With $\theta = 0$ in (3.16) and following we mean that the *difference* between two rapidity is zero.

where $Q(\theta)$ is a function of θ. So the unitarity property becomes

$$f(\theta)f(-\theta) = Q(\theta). \tag{3.32}$$

The solution is a real meromorphic function for θ purely imaginary and it is fixed up to an almost arbitrary function $\phi(\theta)$ which satisfies

$$\begin{aligned} &\phi(\theta) = \phi(i\pi - \theta); \\ &\phi(\theta)\phi(-\theta) = 1. \end{aligned} \tag{3.33}$$

The solution for (3.33) is

$$\phi(\theta) = \prod_{j=1}^{r} \frac{\sinh\theta + i\sinh w_j}{\sinh\theta - i\sinh w_j}, \tag{3.34}$$

where w_j are arbitrary real parameters and r is a natural number from which depends the solution (it's not the total particle number). Equations (3.29) and (3.32) have more than one solution, but among them we can find those having the minimum number of poles and zeros. We call this function $f_{min}(\theta)$. The general solution for (3.29) and (3.32) is

$$f(\theta) = f_{min}(\theta)\phi(\theta) \tag{3.35}$$

and the fact that we can add as many factors as we want in in $\phi(\theta)$ is called CDD-ambiguity.

3.2.4 Elastic scattering.

If we return to selection rules, we can read in point 1 that it is possible that particles among Out-stats are different from those in In-states. If this does not happen, we call the scattering an *elastic scattering.* We can sketch it as

$$\begin{gathered} A_{a_1}(p_1) + \cdots + A_{a_M}(p_M) \to A_{a_i}(p'_1) + \cdots + A_{a_M}(p'_M), \\ \{p_1, \cdots, p_M\} = \{p'_1, \cdots, p'_M\}. \end{gathered} \tag{3.36}$$

Elastic scattering produce a diagonal S-matrix if the mass spectrum is non-degenerate. There is one case of diagonal S-matrix with degenerate mass spectrum, where particles with the same mass $m_i = m_j$ have $\hat{\eta}_i^{\pm} \neq \hat{\eta}_j^{\pm}$[9].
In both cases we say that *reflection*[10] doesn't happen but only *transmission.* In other words

$$S_{ab}^{cd}(\theta_{ab}) = S_{ab}^{ab} \equiv S_{ab} \tag{3.37}$$

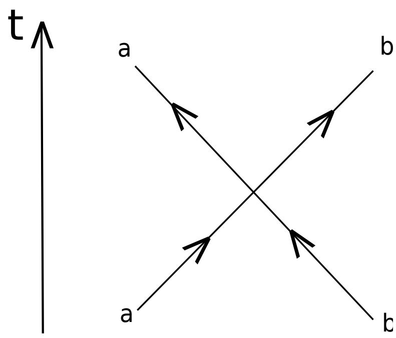


Figure 3.8: Reflection scattering between a and b particles. Time flows from below.

for each elements of the S-matrix. See Fig.(6.41).

We can't have blocks. We can rewrite the unitarity property

$$S_{ab}(\theta)S_{ab}(-\theta) = \mathbb{I} \tag{3.38}$$

and the crossing symmetry property

$$S_{\bar{a}b}(\theta) = S_{a\bar{b}}(\theta) = S_{ab}(i\pi - \theta). \tag{3.39}$$

If we have a non-degenerate mass spectrum we must write

$$A_i(\theta_1)A_j(\theta_2) = S_{Tij}(\theta_{12})A_j(\theta_2)A_i(\theta_1), \tag{3.40}$$

where T means transmission, while if we have a degenerate one we can have reflection

$$A_i(\theta_1)A_j(\theta_2) = S_{Tij}(\theta_{12})A_j(\theta_2)A_i(\theta_1) + S_{Rij}(\theta_{12})A_i(\theta_2)A_j(\theta_1) \tag{3.41}$$

only if particles are distinguishable ($\hat{\eta}_i^{\pm} \neq \hat{\eta}_j^{\pm}$ for $m_i = m_j$). Here R means reflection.

Amplitudes are limited in the momenta, for $p \to \infty$. The more general solution has the shape

$$f(\theta) = \prod_{\alpha \in D} f_\alpha(\theta), \tag{3.42}$$

where $D \subset \mathbb{C}$ invariant by complex conjugation and

$$f_\alpha(\theta) = \frac{\sinh\left(1/2(\theta + i\alpha\pi)\right)}{\sinh\left(1/2(\theta - i\alpha\pi)\right)}. \tag{3.43}$$

[9] $\eta_i^{\pm}$ is the eigenvalue of the operator $Q_i^{\pm}$ in Appendix B.

[10] For instance, when the ingoing particles (a,θ_a) and (b,θ_b) became the Outgoing particles (b,θ_a) and (a,θ_b).

These functions are determined only by the assumptions of being *polynomially bounded* in the momenta or, in rapidity language, functions which are products of exponential like $\exp(ia\sinh([2n+1]\theta))$, $a \in \mathbb{R}$, $n \in \mathbb{N}$, are forbidden. We shall assume that all poles occur on the imaginary θ-axis, i.e. there are no unstable particles. Then every $\alpha \in \mathbb{R}$ and we can choose $-1 < \alpha \leq 1$. Note that (3.42) has a simple pole with residue $2i\sin(\alpha\pi)$ at $\theta = i\alpha\pi$ and a simple zero at $\theta = -i\alpha\pi$. It is possible to find other useful properties from which, if at least one between a or b is real, we find that S_{ab}, up to a sign, must be a product of functions of the form

$$\begin{aligned} F_\alpha(\theta) &= f_\alpha(\theta) f_\alpha(i\pi - \theta) = \\ &= \frac{\sinh\theta + i\sin(\alpha\pi)}{\sinh\theta - i\sin(\alpha\pi)} = \frac{\tanh(\frac{1}{2}(\theta + i\alpha\pi))}{\tanh(\frac{1}{2}(\theta - i\alpha\pi))}. \end{aligned} \tag{3.44}$$

These functions satisfy

$$\begin{aligned} &F_\alpha(\theta) = F_{\alpha+2}(\theta) = F_{1-\alpha}(\theta) = F_{-\alpha}(-\theta), \\ &F_\alpha(\theta) F_{-\alpha}(\theta) = 1, \\ &F_\alpha(\theta - i\pi\beta) F_\alpha(\theta + i\pi\beta) = F_{\alpha-\beta}(\theta) F_{\alpha+\beta}(\theta), \\ &F_0(\theta) = 1. \end{aligned} \tag{3.45}$$

When $0 < \alpha < 1/2$, $F_\alpha(\theta)$ has simple poles at $i\alpha\pi$ and $i(1-\alpha)\pi$ of residues $2i\tan\alpha\pi$ and $-2i\tan\alpha\pi$, respectively, as well as zeros at $-i\alpha\pi$ and $-i(1-\alpha)\pi$. $F_{1/2}(\theta)$ has a double pole at $i\pi/2$ and a double zero at $-i\pi/2$. Obviously, if $1/2 < \alpha' \leq 1$, then $(1-\alpha')$ terms can be treated like $0 < \alpha < 1/2$ terms. We note also that if we know the solution for $(\alpha > 0, \theta > 0)$, from (3.45) we can reconstruct know the solution for $(\alpha < 0, \theta < 0)$.

The poles of a purely elastic S-matrix (paired with zeros via the unitarity condition (3.38)) encapsulate the dynamics of the theory. In fact, poles of each S-matrix element in the physical strip specify uniquely the building blocks f_α into which this S-matrix element factorizes. There aren't redundant poles.

3.2.5 Bootstrap approach for elastic scattering.

We assume that the S-matrix between a bound state and an "elementary" particle can be decomposed in the product of the S-matrix between single particles that constitute the bound state and the "elementary" particle itself. Bound states are in correspondence with poles of the S-matrix. We assume that the *bootstrap principle* holds: bound states are on the same footing as the asymptotic states.

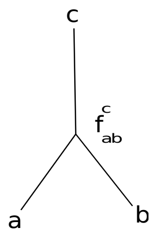


Figure 3.9: a, b and c join in the same vertex of constant structure f^c_{ab}.

Suppose we start with M particles and among them some is elementary and some is a bound state, depending on suitable *fusion rules*.
The choice of whatever particle would became elementary is arbitrary and it is called *nuclear democracy*. For this reason we formulate the *bootstrap consistency principle*: amplitude never changes changing the "nature" of the particles.

Take c as a bound state of a and b[11]. S_{ab} has a pole in $\theta = iu^c_{ab}$ and we can write from (3.25) and (3.37) that

$$S_{ab} \sim \frac{iF^c_{ab}}{\theta - iu^c_{ab}} \tag{3.46}$$

with obvious notations. We take f^c_{ab} completely symmetric, in order to ensure the arbitrariness of the particles nature (also $f^c_{ab} = f^{\overline{c}}_{\overline{a}\overline{b}}$) (Fig.(3.9)).

The mass of the particle c is

$$m_c^2 = s = m_a^2 + m_b^2 + 2m_a m_b \cos u^c_{ab} \tag{3.47}$$

and the same is for a and b. We have obtained the Carnot principle[12]

$$u^c_{ab} + u^b_{\overline{c}a} + u^a_{b\overline{c}} = 2\pi \qquad\qquad \left|u^i_{jk} < \pi\right|. \tag{3.48}$$

See Fig.(3.10).

Now the bootstrap approach comes to light: we want to see the scattering between c and d like the scattering between a,b and d and $S_{cd} \sim S_{ad}S_{bd}$.
The bound state c is defined to be

$$\begin{gathered} |c(\theta)\rangle \equiv \lim_{\nu\to 0} \left|a\big(\theta + i\overline{u}^b_{\overline{c}a} - \nu/2\big)b\big(\theta - i\overline{u}^a_{b\overline{c}} + \nu/2\big)\right\rangle; \\ \overline{u}^k_{ij} \equiv \pi - u^k_{ij}. \end{gathered} \tag{3.49}$$

[11]we take a and b real for simplicity.
[12]For the interested reader, we strongly recommend the Falcioni's paper [51] on the life and the deeds of Lazare Carnot and more.

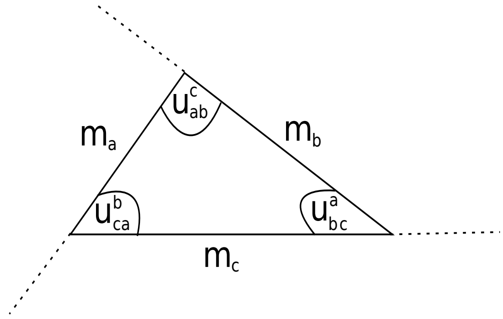


Figure 3.10: Graphic representation of the Carnot rule (3.48).

If we contract $|c\rangle$ with $\langle d|$ we obtain what we were looking for:

$$S_{cd}(\theta) = S_{ad}\big(\theta + i\overline{u}^{b}_{\overline{c}a}\big) S_{bd}\big(\theta - i\overline{u}^{a}_{b\overline{c}}\big). \tag{3.50}$$

This is the *bootstrap equation*.

With the term fusion rule of two particle states we stand for the possible bound states composed by those particles. In general

$$A_i(p_i) \times A_j(p_j) = \sum_k N^k_{ij} A_k(p_i + p_j), \tag{3.51}$$

where $N^k_{ij} = 1$ or $N^k_{ij} = 0$ whether A_k is or is not a permitted bound state of the ingoing two particles. Obviously $N^k_{ij} = 1$ is completely symmetric, like the coupling constants.

Now, if we would know the Lagrangian formulation of our theory, we shall be able to build an exact expression for conserved local charges Q_s, with s the charge spin, which are not Lorentz charges. We recall that these charges derive from local conserved currents generated from some non trivial symmetry which is different from Lorentz symmetry (App. B). If the model is integrable, we could manage infinite local conserved currents and so infinite local conserved charges. We also recall that $s \geq 2$ from [40].
From the S-matrix we can find powerful constraints on the theory, as we shall see now. Then, from Inverse Scattering Method it is possible, in principle, to make up the complete theory and to find the Lagrangian, although it doesn't exist a general way to do this.

In App. B we have defined Q_s as

$$Q'_{(s)} = \Lambda^s Q_{(s)}, \tag{3.52}$$

with no indices summation. We have that

$$
\begin{aligned}
Q_s \left| A_{a_1}(\theta_1) \cdots A_{a_M}(\theta_M) \right\rangle = \sum_{i=1}^{M} w_s^i \left| A_{a_1}(\theta_1) \cdots A_{a_M}(\theta_M) \right\rangle = \\
\sum_{i_1}^{M} \gamma_s^i \exp\left(s\theta_i\right) \left| A_{a_1}(\theta_1) \cdots A_{a_M}(\theta_M) \right\rangle ,
\end{aligned}
\tag{3.53}
$$

where we have made evident the dependence $w(\theta)$ in the term $\exp s\theta$. Using (3.49) we find

$$
w_s^c(\theta) = \gamma_s^c \exp\left(s\theta\right) = \gamma_s^a e^{s(\theta + i\overline{u}_{\overline{c}a}^b)} + \gamma_s^b e^{s(\theta - i\overline{u}_{b\overline{c}}^a)} \tag{3.54}
$$

from which we find the bootstrap consistency equation

$$
\gamma_s^c = \gamma_s^a e^{is\overline{u}_{\overline{c}a}^b} + \gamma_s^b e^{-is\overline{u}_{b\overline{c}}^a}. \tag{3.55}
$$

The condition that this equation holds for all a, b and c such that $f_{ab}^c \neq 0$, and $\gamma_s^a \neq 0$ for at least one a, is a necessary and sufficient condition for the existence of a local integral of motion of spin s, i.e. Q_s.[13]

3.3 S-matrix for $O(n)$ NLSM.

Starting from a very general point of view, Zamolodchikov and Zamolodchikov ([37]) obtained general solutions for the $O(n)$-symmetric factorizable S-matrix. They also found that this S-matrix corresponds to the $O(n)$ NLSM quantum field theory. We follow their derivation for $n \geq 3$, omitting for the moment the $n = 2$ case, corresponding to the Quantum Sine-Gordon model.

3.3.1 Relativistic S-matrix with $O(n)$-symmetry, general solution.

We treat now the class of relativistic factorized S-matrices characterized by the isotopic $O(n)$ symmetry. To introduce this symmetry we assume the existence of $O(n)$-invariant n-vectors representing n particle states $A_i = 1, 2, \cdots, n$ with equal masses m and require the $O(n)$ symmetry of the two-particle scattering (this ensures $O(n)$ symmetry of the total S-matrix due to the factorization). We call these vectors *isospin vectors*. The spectrum of this

[13] If we take $s = \pm 1$, we can find that $Q_{\pm 1}$ correspond to, respectively, to $p = m_i e^{\theta_i}$ and $\overline{p} = m_i e^{-\theta_i}$, so $\gamma_{\pm 1}^i = m_i$.

scattering theory is composed by n particles of mass m. Namely, we assume for two-particles S-matrix the form:

$$S_{ik}^{jl} = \left\langle A_j(p_1')A_l(p_2') \middle| A_i(p_1)A_k(p_2) \right\rangle =$$
$$\delta(p_1 - p_1')\delta(p_2 - p_2')\left[\delta_i^k \delta_j^l S_A(\theta) + \delta_i^j \delta_k^l S_T(\theta) + \delta_i^l \delta_j^k S_R(\theta)\right] \pm (p_1 \leftrightarrow p_2; i \leftrightarrow k)\,, \tag{3.56}$$

where $s = (p_1+p_2)^2$ and $\theta = \theta_1 - \theta_2$, with $\theta_1 > \theta_2$. The $+(-)$ refers to bosons (fermions). The functions S_T and S_R are the transition and reflection amplitudes, respectively, while S_A describes the "annihilation" type processes: $A_i + A_i \;\rightarrow\; A_j + A_j, \quad (i \neq j)$.

The S-matrix (3.56) will be cross-symmetric provided the amplitudes $S(s)$ satisfy equations $S_T(s) = S_T(4m^2 - s)$ and $S_A(s) = S_R(4m^2 - s)$. Dealing with rapidity, crossing-symmetry relations become

$$\begin{aligned} S_T(\theta) &= S_T(i\pi - \theta); \\ S_A(\theta) &= S_R(i\pi - \theta). \end{aligned} \tag{3.57}$$

To describe now the factorized total S-matrix let us introduce, following the general method of section 3.2.1, symbols $A_i, \quad i = 1, 2, \cdots, n$. The commutation rules (3.15) corresponding to (3.56) are

$$\begin{aligned} A_i(\theta_1)A_j(\theta_2) = \delta_{ij} S_A(\theta) \sum_{q=1}^{n} A_q(\theta_2)A_q(\theta_1) + \\ + S_T(\theta)A_j(\theta_2)A_i(\theta_1) + S_R(\theta)A_i(\theta_2)A_j(\theta_1). \end{aligned} \tag{3.58}$$

It is straightforward to obtain the unitarity conditions for two-particle S-matrix (3.56)

$$\begin{aligned} &S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta) = 1; \\ &S_T(\theta)S_R(-\theta) + S_T(-\theta)S_R(\theta) = 0; \\ &nS_A(\theta)S_A(-\theta) + S_A(\theta)S_T(-\theta) + \\ &+ S_A(\theta)S_R(-\theta) + S_T(\theta)S_A(-\theta) + S_R(\theta)S_A(-\theta) = 0. \end{aligned} \tag{3.59}$$

Equations (3.57) and (3.59) are not sufficient to determine the functions $S(\theta)$. Further restrictions arise from (3.14). One can obtain the factorization equations considering all possible three-particles in-products (In-states) $A_i(\theta_1)A_j(\theta_2)A_k(\theta_3)$, reordering them to get out-products (Out-states) by means of (3.58) and requiring the results obtained in two possible successions of two-particles commutations to be equal. The equations arising are evidently different for the cases $n = 2$ and $n \geq 3$ (fewer different three-particle products are possible at $n = 2$). Therefore it is convenient to make a notational distinction between these two cases. Dealing with the case $n = 2$ we denote the amplitudes S_A, S_T and S_R, by σ_A, σ_T and σ_R respectively, reserving the original notations for the case $n \geq 3$.

The factorization equations have the form (the derivation is straightforward but somewhat cumbersome):

1. for n=2

$$\begin{aligned}
&\sigma_T\sigma_A\sigma_R + \sigma_T\sigma_R\sigma_R + \sigma_R\sigma_R\sigma_T = \sigma_R\sigma_T\sigma_R + \sigma_A\sigma_T\sigma_R + \sigma_A\sigma_A\sigma_T;\\
&\sigma_R\sigma_A\sigma_R + \sigma_R\sigma_T\sigma_R = \sigma_R\sigma_R\sigma_A + \sigma_R\sigma_R\sigma_T + \sigma_T\sigma_R\sigma_A + \sigma_T\sigma_R\sigma_R +\\
&+ 2\sigma_A\sigma_R\sigma_A + \sigma_A\sigma_R\sigma_T + \sigma_A\sigma_R\sigma_T + \sigma_A\sigma_R\sigma_R + \sigma_A\sigma_T\sigma_A + \sigma_A\sigma_A\sigma_A;
\end{aligned} \tag{3.60}$$

2. for $n \geq 3$

$$\begin{aligned}
&S_TS_RS_R + S_RS_RS_T = S_RS_TS_R;\\
&S_TS_AS_A + S_RS_TS_A = S_RS_AS_T;\\
&nS_AS_RS_A + S_AS_RS_T + S_AS_RS_R + S_AS_TS_A +\\
&+ S_TS_RS_A + S_RS_RS_A + S_AS_AS_A = S_RS_AS_R.
\end{aligned} \tag{3.61}$$

For each term in (3.60) and (3.61) the argument of the first, the second and the third S (σ in (3.60) is implied to be θ, $\theta' + \theta$ and θ' respectively.

The factorization equations turn out to be rather restrictive. They allow one to express explicitly all the amplitudes in terms of one function.
General solutions for both systems (3.60) and (3.61) satisfying the real-analyticity condition are derived in [37]. We concentrate now on $n \geq 3$ theory.

The general solution for (3.61) contains only one free parameter λ and has the form:

$$\begin{aligned}
S_R(\theta) &= -\frac{i\lambda}{\theta} S_T(\theta);\\
S_A(\theta) &= -\frac{i\lambda}{i[(n-2)/2]\lambda - \theta} S_T(\theta).
\end{aligned} \tag{3.62}$$

The restrictions on the amplitude $S_T(\theta)$ come from the unitarity conditions (3.59). The second and the third of these equations are satisfied by (3.62) identically, while the first gives

$$S_T(\theta)S_T(-\theta) = \frac{\theta^2}{\theta^2 + \lambda^2}. \tag{3.63}$$

Until now we have deliberately avoided the use of the crossing-symmetry relations. Although the above consideration concerns the relativistic case, the unitarity conditions (3.59) and factorization equations (3.61) are valid for any non- relativistic $O(n)$ symmetric factorized S-matrix as well, under the substitution:

$$\theta \to \frac{k}{m} = \frac{k_1 - k_2}{m}, \tag{3.64}$$

where k_1 and k_2 are momenta of the colliding particles. Therefore, the general solutions (3.62) and (3.63) are still valid (after the substitution (3.64) in a nonrelativistic case.

Equations (3.57) give restrictions on free parameters in (3.62). It is easy to see that (3.57) is satisfied only if

$$\lambda = \frac{2\pi}{n-2}. \tag{3.65}$$

Thus, the formulas for $n \geq 3$ do not actually contain any free parameter.

The first equation in (3.57) together with (3.63) will be used to determine $S_T(\theta)$. The solution admits the CDD-ambiguity only: an arbitrary solution can be obtained multiplying some "minimum" solution by a meromorphic function $f(\theta)$ (return to (3.34)).
In the case $n \geq 3$ there are, in general, two different "minimum" solutions (the exceptional cases are $n = 3, 4$, when these two solutions coincide). We denote these solutions $S_T^+(\theta)$ and $S_T^-(\theta)$; they can be written in the form

$$S_T^{\pm} = Q^{\pm}(\theta) Q^{\pm}(i\pi - \theta), \tag{3.66}$$

where

$$Q^{\pm}(\theta) = \frac{\Gamma\left(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)}. \tag{3.67}$$

In the following Sections we point out the relation between the solutions (3.66) and certain two-dimensional quantum field theory models. Namely, we show that the solutions (3.66) give the exact S-matrices for the NLSM and for the "fundamental" fermions of Gross-Neveu model[14], respectively.

3.3.2 $O(n)$ NLSM with $n \geq 3$.

We want to find a field theory, so a dynamical background, which reproduces the S-matrix (3.56), solved by (3.62) and (3.66), (3.67). It is better to remember the fundamental properties another time:

[14]We mention it only for completeness. we shall discuss only NLSM.

a) we deal with real massive particles, which are organized in isovectors *n-plets*[15];

b) the correspondent S-matrix is factorized.

we shall find that the Lagrangian which represent the correspondent dynamical quantum field theory is the Lagrangian of a NLSM, like that in (2.11). We reproduce it here, with some difference for notation convenience

$$\mathscr{L} = -\frac{1}{2g_0}\sum_{i=1}^{n} n_{i,\mu}^2 \qquad n^2 = 1, \tag{3.68}$$

where $n \equiv \vec{a}$ and g_0 is *the only one* (bare) coupling constant.

We already have the exact solution of the S-matrix but we don't know the correspondent field theory. So we can start for an ansatz theory like (3.68) and, via perturbation expansion of its generating function, find the respective Feynman rules. But the *normal* g-expansion (expansion using function of the coupling constant like coefficient) brings us to infrared divergences. So we use another useful method, named the $1/n$ *expansion*. This method is based upon the fact that interaction amplitudes are of the order $1/n$ and so our expansion is made with respect to this interactions. We shall find the solution for the $n \to \infty$ limit and we assume that results work also for little n. This section confirms this assumption to some extent.

We have to introduce the auxiliary Lagrange field (an example can be found in section 2.2) $w(x)$ in (3.68)

$$\mathscr{L} = -\frac{1}{2g_0}\sum_{i=1}^{n}\left[n_{i,\mu}^2(x) + w(x)n_i^2\right] - \frac{w(x)}{2g_0}; \tag{3.69}$$

the generating function is then

$$\begin{aligned} Z[J] = \int \prod_x \left[dw(x)\prod_i dn_i(x)\right] \times \\ \times \exp\left\{ i\int d^2x \left[\mathscr{L}[n_i, w] + g_0^{1/2} J^i(x) n_i(x)\right]\right\} \end{aligned} \tag{3.70}$$

[15] n-vectors invariant by $O(n)$ transformations.

Integration with respect to n_i gives

$$Z[J_i] = \int \prod_x w(x) \exp\left\{iS^{\text{eff}}[w]\right\} \times$$
$$\times \exp\left\{\frac{i}{2}\int d^x d^2x' J_i(x)J_i(x')G(x,x'|w)\right\} \tag{3.71}$$

where

$$S^{\text{eff}}[w] = i\frac{n}{2} tr \ln(\Box - w(x)) - \int d^2x \frac{w(x)}{2g_0},$$

$G(x,x'|w)$ is the Green function of $\Box - w(x)$.

We obtain the $1/n$ expansion of (3.69) calculating (3.71) perturbatively, using the stationary phase method. The saddle point is

$$w'(x) = \Lambda^2 \exp\left\{-\frac{4\pi}{ng_0}\right\}, \tag{3.72}$$

where Λ is an ultraviolet cut-off. We expand (3.71) around $\overline{w}(x) = w(x) - w'(x)$ and finally we obtain the following Feynman rules: Fig.(3.11).

$$\begin{aligned} G(k) &= \frac{i}{k^2 - m^2 + i\epsilon}, \\ D(k^2)^{-1} &= \frac{1}{4\pi^2}\int \frac{dp^2}{[p^2 - m^2 + i\epsilon][(p+k)^2 - m^2 + i\epsilon]}. \end{aligned} \tag{3.73}$$

Starting from section 2.3.3, we shall see that only the first two conservation laws (2.59) are already sufficient to restrict the S-matrix to the processes satisfying the selection rules 1) and 2) in section 3.2. According to what we have said before, this implies the S-matrix factorization for model (3.69).

Now, we want to go deep into the detail of the S-matrix: we shall see that the production of particles is not possible, working out the amplitude for $2 \to 4$ particles. It is possible to see that the $1/n^2$ terms cancel all possible contributes from $1/n$ $2 \to 4$ diagrams. We see in Fig.(3.12) the complete $2 \to 4$ diagram and in Fig.(3.13) all possible *particular* diagrams. We choose for simplicity diagrams with $i \neq j \neq k$.

It has been turned out that an arbitrary bosonic loop is the sum of term, each of them corresponding to one specific division of the loop, like in Fig.(3.14). The contribution of each division is equal to the product of

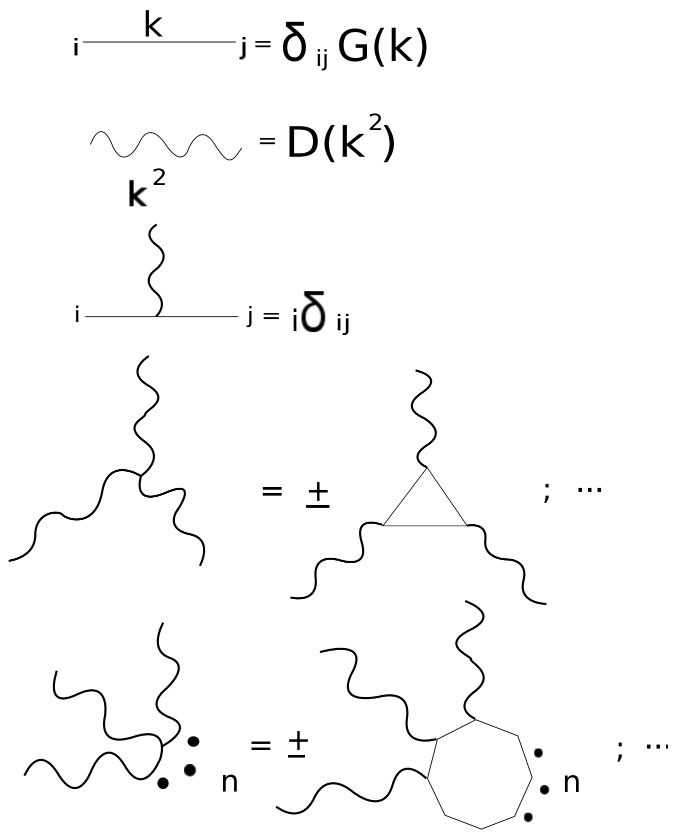


Figure 3.11: Elements of the $1/N$-diagrammatic technique for $O(N)$ NLSM.

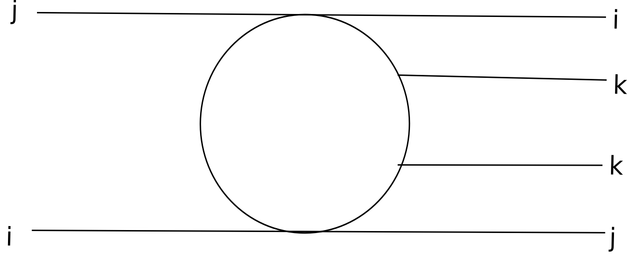


Figure 3.12: $2 \to 4$ scattering amplitude with a double k-particle production.

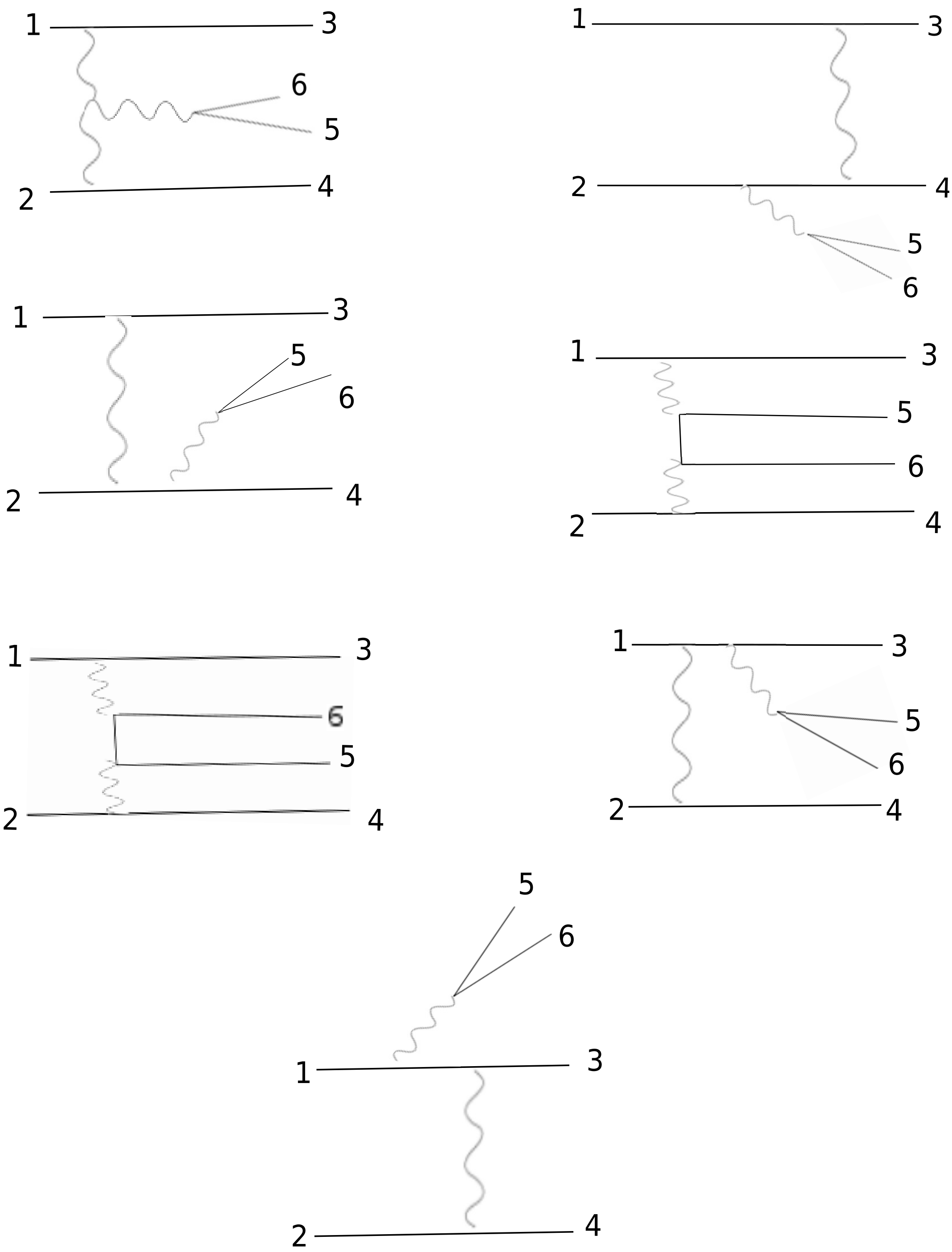


Figure 3.13: $2 \to 4$ amplitudes. Here numbers are particles label.

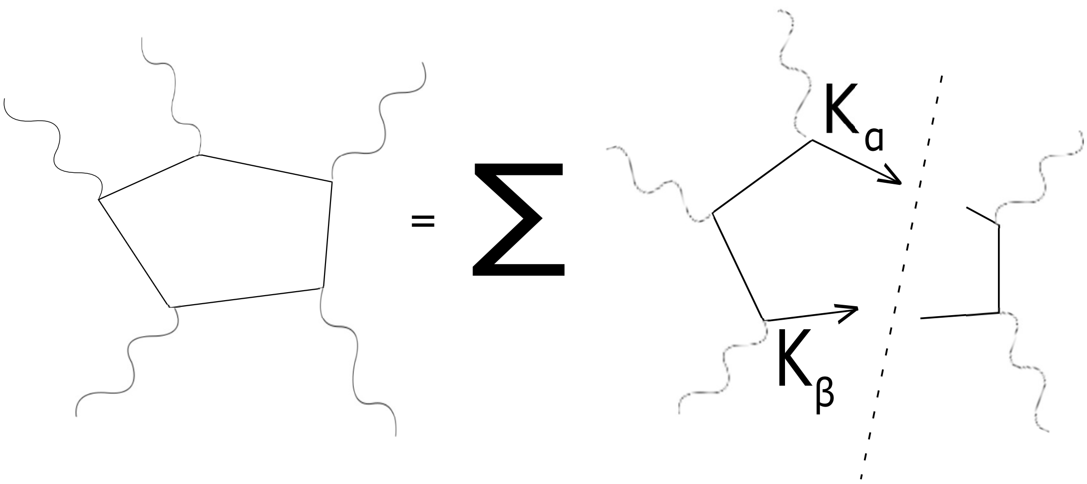


Figure 3.14: The "division rule" for the calculation of an arbitrary one-loop diagram.

two "tree" diagrams separated by a dashed line in the same figure by the function

$$i\phi(s_{ab}) = \frac{1}{4\pi^2}\int \frac{dp}{[p^2 - m^2 + i\epsilon][(p + k_a + k_b)^2 - m^2 + i\epsilon]}, \tag{3.74}$$

if $s_{ab} = (k_a + k_b)^2$ and $k_a^2 = k_b^2 = m^2$. At s_{ab} fixed, this equation has two solutions connected by the exchange $k_a \leftrightarrow k_b$.

For instance, we consider the $1/n^2$ diagram consistent of a triangle-loop, like in Fig.(3.15). We have two solutions corresponding to $k_1^2 = k_2^2 = m^2$:

1. $(k_1, k_2) = (p_1, p_3)$,

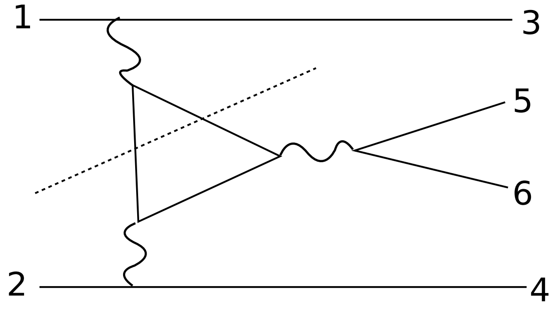


Figure 3.15: One of the three possible divisions of the 1-loop 3-vertex for the $2 \to 4$ scattering.

2. $(k_1, k_2) = (p_3, p_1)$.

We have for this single division

$$S^{3456}_{12_{|a(b)}} \times i\phi(s_{13(31)}) \times D(s_{13(31)}). \tag{3.75}$$

But the solution for (3.74) in this case is

$$i\phi(s_{13(31)}) = -\frac{1}{D(s_{13(31)})} \tag{3.76}$$

We can easily check now that the contribution of this division is the opposite of the last two diagram contributions in Fig.(3.13), so these amplitudes are cancelled by (3.75) $1/n^2$-contribution.
In the same way we could turn out that $3 \to 3$ amplitude are different from zero if the S-matrix is factorizable.

We have just verified that S-matrix (3.56) with solution for $n \geq 3$ is compatible with (3.68) theory, because of (3.68) satisfies a) and b) rules.

We can add more considerations. First, we have to deal with CDD-ambiguity. In fact, in principle we don't know the exact S-matrix, due to the ambiguous contribution of (3.34). In fact, if we add the so called *CDD poles*[16], they result added in all three channels of two-particles scattering (s,t and u), but if we choose the minimum solution $S_T(\theta) = S^{\pm}_T(\theta)$.

We shall see now that the choice of a solution of kind $f(\theta) = f_{\min}(\theta)$ without CDD-poles for the scattering of particles in the theory (3.68) is supported by the $1/n$-expansion.
In fact, take $S_T(\theta) = S^{+}_T(\theta)$. We don't have any pole in the physical strip, so it is impossible to have bound state. Calculation of two-particle amplitudes for model (3.68) by $1/n$-expansion technique in the order of $1/n$ leads to the result written in Fig(3.16). We can easily see that, for $n \to \infty$, $S_R(\theta)$ and $S_A(\theta)$ go to zero, that means: *bound states are unlikely.*

[16]the arbitrary numbers in (3.34) named w_j.

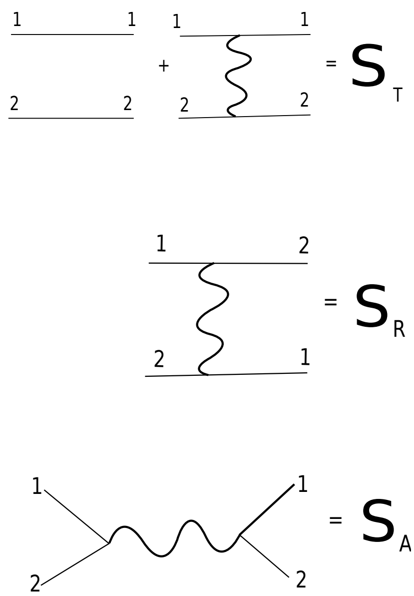


Figure 3.16: Graphic representation of the $O(3)$ NLSM S-matrix elements.

We have just verified that A_i particle state in this model doesn't give any bound state and that the solution of the 2dNLSM S-matrix for $n \geq 3$ is

$$\begin{cases} S_R(\theta) = -\dfrac{i\lambda}{\theta} S_T(\theta); \\ S_A(\theta) = -\dfrac{i\lambda}{i[(n-2)/2]\lambda - \theta} S_T(\theta); \\ S_T(\theta) = Q^+(\theta) Q^+(i\pi - \theta); \\ \\ Q^+(\theta) = \dfrac{\Gamma\left(+\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)}; \\ \lambda = \dfrac{2\pi}{n-2}. \end{cases} \tag{3.77}$$

It is easy to verify that expressions Fig.(3.16) really coincide with the first terms of $1/n$-expansion of exact solution (3.77). Thus, the latter choice is in accordance with $1/n$-expansion of (3.68). Finally it is worth remembering that we have assumed the validity of $1/n$-expansion also for $n \geq 3$.

Chapter 4

Thermodynamic Bethe Ansatz.

The Thermodynamic Bethe ansatz is a very powerful method to calculate the thermodynamics (and, as we shall see, the finite-size effects) of a relativistic theory in two dimensions. The only two ingredients we need are the S-matrix and the mass spectrum of the particles. We have already seen that some integrable theories have a completely defined S-matrix. With TBA we shall find the finite-size *scaling coefficient* $\tilde{c}$ and its values in the UV and IR limits (small and large length scale, respectively) for *purely elastic scattering theories*[1] and general scattering theories, applying the Bethe Ansatz Technique.

TBA is a very powerful tool for the computation of Energy, Free Energy and the other thermodynamic quantities. Moreover, TBA is useful to inspect the UV regime of an integrable QFT, in order to find possible relations with some CFT. In fact, we shall see that, perturbing with "good" operators a specific CFT, it is possible to have a still integrable quantum field theory based on a fundamental mass scale.

In the first section we quickly summarize important aspects of 2-dimensional CFT. In the second section we find the Bethe Ansatz equations for a relativistic system of N particles of different species. In the third section we find the TBA equations, the values of $\tilde{c}$ and of the ground-state energy. We find the precise asymptotic values of $\tilde{c}$ in the two IR and UV limits.

In the fourth section we take a look to perturbed CFT and to the intriguing coding of TBA through Lie Algebra Dynkin diagrams. TBA equations can be rewritten in a more universal form as functional equations, called Y-system, where $Y(\theta)$'s are functions of the rapidity θ. Such functional equations posses a peculiar periodicity that can be related to the conformal dimensions of the relevant operator perturbing the UV CFT into a full QFT with scale. More-

[1]A purely elastic scattering theory has a S-matrix which presents only transmission scattering elements.

over, assuming different analytic properties for the $Y(\theta)$ functions, one can reconstruct the scaling functions not only for the vacuum, but for all excited states. Thus, the Y-system codes the full Hilbert space of an integrable QFT and its determination is a fundamental step in understanding integrable QFT fully.
Section 4.5 contains a brief review on quantum integrability based upon the concept of R-matrix, transfer matrix, spin chains, quantum groups, algebraic Bethe ansatz. Everything is briefly introduced. After that, we find the TBA equations for the Sine-Gordon model in their universal form.
Eventually, we find the exact CFT, UV limit of the $O(3)$ NLSM.

4.1 Some features of 2d-Conformal Field Theory.

In this section we focus our attention on some aspect of 2d-CFT. We refer the interested reader to [17] and to [18] for a closer examination.
CFTs are defined as those invariant under conformal transformations of the metric

$$g_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{4.1}$$

where $\Lambda(x)$ is an arbitrary smooth scalar function of the coordinates. The name *conformal* derives from the property of conservation of the angle between two curves crossing each other at some point of the space time. A field theory is conformal if it's invariant under

1. translation $\quad x'^{\mu} = x^{\mu} + a^{\mu}$;
2. dilation $\quad x'^{\mu} = \alpha x^{\mu}$;
3. rigid rotation $\quad x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$.
4. special conformal transformation $\quad x'^{\mu} = \frac{x^{\mu} - b^{\mu}\mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2}$

where a_μ is free of constraints and $b_\mu \equiv c^{\sigma}{}_{\sigma\mu}/d$, with $c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho + -\eta_{\nu\rho} b_\mu$ in a d-dimensional spacetime.
In 2d, it is more effective to consider complex coordinates like

$$z = x^0 + i x^1, \qquad \overline{z} = x^0 - i x^1, \tag{4.2}$$

because a conformal transformation in these coordinates can be seen as a complex mapping $z \to w(z)$ and $\overline{z} \to \overline{w}(\overline{z})$, i.e.

$$dw = \left(\frac{dw}{dz}\right) dz \tag{4.3}$$

where dw/dz contains both dilation factor $|dw/dz|$ and a phase $\arg(dw/dz)$, which embodies a rotation (the same for $\overline{z}$).

The importance of 2d CFTs is double. First, they are completely solvable problems, because the algebra of the "conformal group" is infinite (it is the *De Witt Algebra*). Second, they describe exactly the string.

A conformal theory is invariant under a change of scale. If a theory doesn't have a mass-scale, it is conformal[2]. But if we introduce a scale, we break the symmetry and the theory is not conformal anymore. For example, every classical sigma model is conformal, but if we quantize it, in general it acquires a mass scale and it looses conformal properties. Nevertheless, we have just seen that there may exist quantum conformal field theories which preserve conformal invariance also at the quantum level.

An important role is played by the energy-momentum tensor $T_{\mu\nu}(x^0, x^1)$. This tensor is traceless if the theory is conformally invariant. In the complex formalism illustrated above, the energy-momentum tensor is "divided" in two distinct parts, the *holomorfic* $T(z)$ and the *anti-holomorfic* $\overline{T}(\overline{z})$, each one depending respectively on z or $\overline{z}$ only.

4.1.1 Central charge.

In QFT, under very general assumptions, the validity of the Wilson short distance *operator product expansion* (OPE) holds

$$A_1(z)A_2(w) \sim \sum_i C^i_{12}(z-w)A_i(w), \tag{4.4}$$

where $A_i(z)$ is a generic field and C^i_{12} is a set of c-functions that diverges in $w = z$. In fact, it is typical of correlation functions to have singularities when the position of two or more fields coincide. This is the manifestation of the infinite fluctuations which a quantum field undergoes in a precise position. OPE is the representation of a product of operators taken in two different positions, z and w, by a sum of terms, each being a single operator regular in $w \to z$, multiplied by a c-function which becomes infinite in $w \to z$ limit. The symbol $\sim$ in (4.4) becomes $=$ only evaluating the expression into correlators:

$$\begin{aligned} A_1(z)A_2(w) &\sim \sum_{n=-\infty}^{N} C^n_{12}\frac{A_n(w)}{(z-w)^n}, \\ \langle A_1(z)A_2(w)\rangle &= \sum_{n=-\infty}^{N} C^n_{12}\frac{A_n(w)}{(z-w)^n}. \end{aligned} \tag{4.5}$$

[2]This is strictly true only if we assume the so-called "Polyakov conjecture" which turns out to be valid classically in all non-pathological cases of relativistically invariant 2d QFT

N can be infinite or finite.
The energy-momentum tensor OPE is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_z T(w)}{z-w}, \tag{4.6}$$

where c is defined to be the *central charge*[3] of the theory and it is dependent from the model. For instance, if we have a theory of free bosons, $c = 1$; free Majorana fermions have $c = 1/2$.

The existence in any field theory of the conformal charge indicates the presence of a "soft breaking" of the conformal invariance by the presence of a *macroscopic* scale in the theory. For instance, if a particular theory is defined on a surface with boundaries, this implies the automatic appearance of c. If we introduce a scale we, de facto, quantize the theory, passing from the De Witt to the *Virasoro algebra* of the generators of the conformal "group":

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n, \overline{L}_m] &= 0, \\ [\overline{L}_n, \overline{L}_m] &= (n-m)\overline{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \end{aligned} \tag{4.7}$$

where

$$\begin{cases} T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \\ \overline{T}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{z}^{-n-2}\overline{L}_n, \\ L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z), \\ \overline{L}_n = \frac{1}{2\pi i}\oint d\overline{z}\overline{z}^{n+1}\overline{T}(\overline{z}). \end{cases} \tag{4.8}$$

Giving an arbitrary infinitesimal conformal coordinate variation $\epsilon^\nu(x)$, it is possible to obtain the so-called *conformal Ward identity* for the generic field collection $X \equiv A_1(z_1)\cdots A_n(z_n)$

$$\delta_{\epsilon,\overline{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint dz\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint d\overline{z}\overline{\epsilon}(\overline{z})\langle\overline{T}(\overline{z})X\rangle, \tag{4.9}$$

where $\delta_{\epsilon,\overline{\epsilon}}\langle X\rangle$ is the variation of X under an infinitesimal local conformal transformation. From (4.9) and (4.6) it is possible to find the variation of

[3] It is also known with the name of *conformal anomaly*.

$T(z)$ under a finite conformal mapping $z \to w(z)$

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right], \tag{4.10}$$

where we have introduced the *Schwarzian derivative*:

$$\{w; z\} = \frac{(d^3w/dz^3)}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2. \tag{4.11}$$

From the properties of (4.11) we find that (4.10) can be written as

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z; w\}. \tag{4.12}$$

There exists conformal mapping which introduces macroscopic scale in the theory. We illustrate this with an important example.

We map a (classical) conformal field theory living on the whole complex plane onto a cylinder of circumference L. L will be the macroscopic scale of the system. The transformation can easily be found as

$$w(z) = \frac{L}{2\pi} \ln z. \tag{4.13}$$

From (4.12), it is easy to find that the corresponding energy-momentum tensor is related to that of the plane by

$$T_{\text{cyl}}(w) = \left(\frac{2\pi}{L}\right) \left[T_{\text{pl}}(z)z^2 - \frac{c}{24}\right]. \tag{4.14}$$

If one assumes that $\langle T_{\text{pl}} \rangle = 0$, then

$$\langle T_{\text{cyl}} \rangle = -\frac{c\pi^2}{6L^2}. \tag{4.15}$$

The vacuum energy of the theory on a cylinder on macroscopic dimension L is not zero and it is proportional to the *Casimir energy* of a quantum field theory. Obviously, if $L \to \infty$, $\langle T_{\text{cyl}} \rangle \to 0$. An important remark: the "true" energy density is not $\langle T(z) \rangle$ but $\langle T^{00}(x^0, x^1) \rangle$, that is

$$\langle T^{00} \rangle = \frac{c\pi}{6L^2}. \tag{4.16}$$

This is a density. The energy can be found by multiplication of $\langle T^{00} \rangle$ by L.

It is possible to relate the variation of the free energy $F(L)$ of the L-system with the variation of the energy momentum tensor and the variation of the metric. It can be easily seen that

$$\delta F(L) = \int d^2w \left(f_0 + \frac{c\pi}{6L^2} \right) \frac{\delta L}{L}, \tag{4.17}$$

where, for generality, we have supposed that $\langle T_{\text{pl}} \rangle \neq 0$ and so we have introduced f_0, the free energy per unit area of the plane or in the $L \to \infty$ limit. After integration it follows that

$$F(L) = f_0 L - \frac{c\pi}{6L}. \tag{4.18}$$

Note that c arises also when the field theory is defined on a curved space. In this case the anomaly appears in the trace of the energy momentum tensor, usually zero:

$$\left\langle T^{\mu}{}_{\mu}(x) \right\rangle = \frac{c}{24\pi} R(x), \tag{4.19}$$

where $R(x)$ is the curvature of the two dimensional manifold. This "metric" breaking is called the *trace anomaly.*

4.1.2 Double periodic boundary conditions. CFT on the torus.

Imposing boundary conditions on one or two variables could modify the space on which fields live. For instance, starting from a plane, when we impose that field must be periodic in the x_0 direction, i.e. $\phi(x_0, x_1) = \phi(x_0 + L, x_1)$, we realize an infinite cylinder as a strip of the plane. Each periodic strip is equivalent to the same cylinder. If we impose periodic boundary conditions also in the x_1 direction, $\phi(x_0, x_1) = \phi(x_0, x_1 + R)$, we pass from the plane to the *torus* (Fig.(4.1)). It is like cutting the infinite cylinder above and below, leaving a finite cylinder of length R. Then we can glue together the extremities and build the torus. In other words, starting from the complex plane we can choose two vectors (w_1, w_2), from which we make the transformation $z = \exp\left[(2\pi i/w_2)w\right]$, obtaining a cylinder. Then, asking a second periodicity, we have the torus. We define a torus by specifying $\tau = w_2/w_1$. The torus is a new kind of topological space with respect to the plane or the cylinder: in fact it has one handle (i.e., it has genus $h = 1$) differently from the plane and the cylinder, which haven't any handle (they have $h = 0$).

If we take one direction of the torus, for instance L, and we impose $L \to \infty$, than we have an infinite cylinder of circumference R and we can

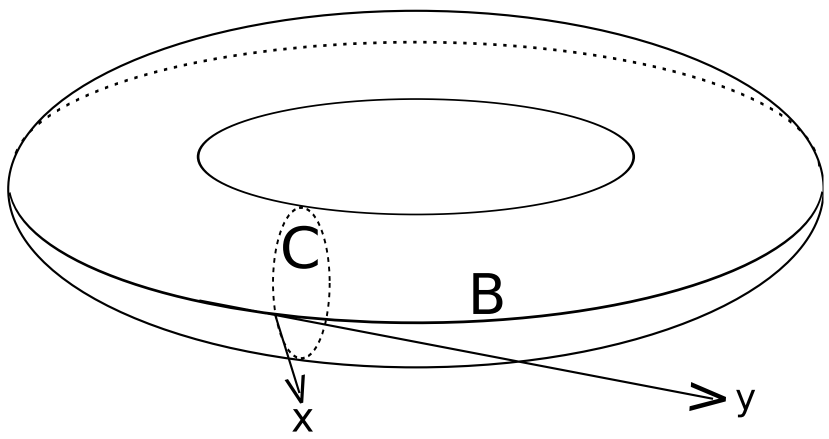


Figure 4.1: Flat torus generated by two orthogonal geodesic circles C and B of circumference R and L respectively.

have $F(R) = f_0 R - \tilde{c}\pi/6R + O(1/R)$. Generally, conformal anomaly is named the *finite-size scaling coefficient* and it is equal to

$$\tilde{c} = c - 12 d_{\min}, \tag{4.20}$$

where c is the central charge of the theory and $d_{\min}$ is the *lowest scaling dimension*[4] of the CFT in consideration. We recall that the ground-state $|\min\rangle$ have the lowest eigenvalue of L_0 (return to (4.7) and (4.8)), that is $L_0|\min\rangle = \Delta_{\min}|\min\rangle$.

$\tilde{c}$ it is not a constant, but it depends (by dimensional arguments) from the dimensionless quantity $r = R/R_c$, where $R_c = 1/M$ is the *largest correlation length*, in other words the maximum length of influence between particles. M is the smallest mass of the theory. We define two sectors

1. UV limit, when $r \to 0$, i.e. $R_c \to \infty$, i.e. $M \to 0$

2. IR limit, when $r \to \infty$, i.e. $R_c \to 0$, i.e. $M \to \infty$

The ground-state energy can be found from (4.16)

$$E_{\min}(R) = -\frac{\tilde{c}_{\min}(r)\pi}{6R}. \tag{4.21}$$

[4] $d_{\min} = \Delta_{\min} + \overline{\Delta}_{\min}$, where the label "min" means "ground state" and Δ is the scaling dimension of the theory. The lowest scaling dimension is, in unitary theories, the vacuum scaling dimension, that is $d_{\min} = 0$. In non unitary theories, it could happen that conformal vacuum and ground state don't coincide.

The ground-state energy (4.21) and the finite-size scaling function $\tilde{c}$ become functions of the scaling parameter $r = MR$. If we expand, we find

$$\begin{aligned}\tilde{c}_{\min}(r) &= \tilde{c} + \tilde{c}_{\min,1} R^{\delta} + \tilde{c}_{\min,2} R^{2\delta} + \cdots = \\ &= \tilde{c} + \tilde{d}_{\min,1} (MR)^{\delta} + \tilde{d}_{\min,2} (MR)^{2\delta} + \cdots = \\ &= \tilde{c} + \tilde{d}_{\min,1} r^{\delta} + \tilde{d}_{\min,2} r^{2\delta} + \cdots\end{aligned} \tag{4.22}$$

where

$$\begin{cases} \delta \in \mathbb{R}, \\ \tilde{c}_{\min,n} \text{ are } M^{n\delta}\text{-dimension coefficients,} \\ \tilde{d}_{\min,n} \text{ are dimension-less coefficients,} \end{cases} \tag{4.23}$$

It is possible to find the values for the excited states i. In this case we must replace 0 by $i \in \mathbb{N}$ in (4.21) and (4.22). We call $r \to 0$ the UV-limit because it means $R \to 0$ but also $M \to 0$, i.e. each energy in the theory is rescaled, growing to infinite values which respect to the fundamental mass M. UV is the high-energy limit.
On the contrary, the IR limit is $r \to \infty$, that is $M \to \infty$. Each mass becomes insignificant with respect to the scale M and all massive states decouple. IR limit is the low-energy limit.
The partition function on the torus takes the form

$$Z(w_1, w_2) = \mathrm{Tr} \exp - \left(H \mathrm{Im} w_2 - i P \mathrm{Re} w_2 \right), \tag{4.24}$$

with H the Hamiltonian and P the Momentum operator. Our target will be to find this energy, together with other thermodynamic variables.

4.2 Relativistic Bethe Ansatz.

We consider a relativistic integrable purely elastic field theory in $(1 + 1)$ dimensions defined on a cylinder of circumference l. In this theory there are n different species of particles. We consider N particles, at positions $x_1, \cdots, x_N$, N_a of which are of species a.
In relativistic theory the wave function formalism is inappropriate to describe a system of relativistic particles (this is due to virtual and real particle creation). However, there exist regions in the configuration space where a set of relativistic particles can be strongly separated in their space positions x_i. More specifically, that means $|x_i - x_j| \gg R_c$. In these regions, which we call *free regions*, the particles move as free ones and off-mass-shell effects can be neglected. If the space coordinates and the momenta of particles are,

respectively, x and p, the wave function of a asymptotic state of N particle is

$$\Psi(x_1,\cdots,x_N)=\left[\prod_{i=1}^{N}e^{ip_ix_i}\right]\times\sum_{Q\in S_N}A(Q)\Theta(x_Q), \tag{4.25}$$

where the second sum runs over the $N!$ permutations $Q\in S_N$ of the N particle positions on the line segment $[0,L]$ and the $A(Q)$ are coefficients depending on particle momenta, whose ordering on the line is specified by

$$\Theta(x_Q)=\begin{cases}1 & \text{if } x_{Q_1}<\cdots<x_{Q_N}\\ 0 & \text{otherwise}\end{cases} \tag{4.26}$$

The coefficients are determined by the S-matrix of the theory.

The number of particles N and the set of their momenta $p_i,\ i=1,2,...,N$ remain the same in all the free regions, thanks to the factorization of the S-matrix. The wave function (4.25) of these states is the so-called *Bethe wave function*. We denote a free region as $\{i_1,i_2,\cdots,i_N\}$ if $x_{i_1}<x_{i_2}<\cdots<x_{i_N}$.

The transition between two adjacent free regions involves configurations where two or more particles are close to each other. In these configurations, of course, the relativistic effects can't be neglected and the wave function formalism is not anymore valid. However, thanks to the S-matrix , we are able to provide conditions in order to write wave functions in adjacency regions. In the purely elastic case every transition, say $\{i_1,\cdots,i_q,i_{q+1},\cdots,i_N\}\to\{i_1,\cdots,i_{q+1},i_q,\cdots,i_N\}$, results in a multiplication of the wave function by the corresponding scattering amplitude, $S(\theta_{i_q}-\theta_{i_{q+1}})$ in this case. Note that for $\theta\in\mathbb{R}$, this amplitude is a number with a unit module, i.e.

$$S(\theta)=\exp i\chi(\theta) \tag{4.27}$$

with real phase $\chi(\theta)$.[5]

The coefficients of the wave function after the scattering of two particles become

$$A(Q')=S_{q,q+1}(\theta_{i_q}-\theta_{i_{q+1}})A(Q) \tag{4.28}$$

if the permutation Q and Q' differs only by the exchange of two elements. We call $q=i$ and $q+1=j$ for sake of simplicity.

Because of the periodicity, for a cylinder with circumference of length l, we find that

$$\Psi(x_1,\cdots,x_i=0,\cdots,x_N)=\pm\Psi(x_1,\cdots,x_i=l,\cdots,x_N), \tag{4.29}$$

[5]We want to remark that the only observable in purely elastic scattering theories is the time delay, compared to the free case. The time delay of a and b depends on the S-matrix only through the rapidity derivative of the phase shift, i.e. $\phi_{ab}=-id/d\theta\ln S_{ab}$.

where $+$ is for bosons and $-$ is for fermions. *This is* the specification of bosons and fermions. We shall see below that bosons and fermions exist in two sectors, the difference between them is the application of the exclusion principle (we shall see these two sectors in the next lines).
(4.29) leads to

$$A(i, Q_2, \cdots, Q_N) = \pm e^{ip_i l} A(Q_2, \cdots, Q_N, i) \tag{4.30}$$

for any $Q \in S_N$ such that $Q_1 = i$. From (4.28) and (4.30) we find the *Bethe ansatz equations* for the particle i

$$l m_i \sinh\theta_i + \sum_{j \neq i} \chi_{ij}(\theta_i - \theta_j) = 2\pi n_i \qquad i = 1, \cdots, N. \tag{4.31}$$

$\chi_{ij} = -i \ln S_{ij}(\theta_i - \theta_j)$ is the phase shift and n_i can be considered to be the quantum numbers of the state of the multi-particle system:

1. $n_i \in \mathbb{Z}$ if the particle is a boson.
2. $n_i \in \mathbb{Z} + \frac{1}{2}$ if the particle is a fermion.

Solutions of this system of transcendental equations are the permitted sets of rapidities $(\theta_1, \cdots, \theta_N)$ in free regions $\{i_1, \cdots, i_N\}$, up to an error related to the goodness of (4.25) approximation.

The energy and momentum of the state $(\theta_1, \cdots, \theta_N)$ for particle i with mass m_i are

$$H = \sum_j^N m_i \cosh\theta_j, \qquad\qquad P = \sum_j^N m_i \sinh\theta_j \tag{4.32}$$

An important role is played by the unitarity condition $S(0)^2 = 1$. This implies that $S(0) = \pm 1$ and leads to the two sectors we have already spoken about

1. $S(0) = -1$. If two particles with the same rapidities scatter, the wave function is antisymmetric. This is incompatible with the Bose statistic, so particles with $n_i \in \mathbb{Z}$ undergo an exclusion principle based on rapidity. We shall denote this situation in the Bethe ansatz equations saying that bosons are particles of *fermionic type*. Fermions, or particles with $n_i \in \mathbb{Z} + \frac{1}{2}$, can have the same rapidity and they are called particles of *bosonic type*.
2. $S(0) = +1$. If two particles with the same rapidities scatter, the wave function is symmetric. Here bosons undergo bosonic scattering and fermions undergo fermionic scattering. Someone can link this two possibilities as a sort of SuSy entanglement.

4.3 Thermodynamic Bethe Ansatz (TBA).

The TBA technique consists in two distinct parts.The first is based on the observation that for a theory with particles involving only purely elastic scattering the asymptotic wave function describing particle states, i.e. the wave function when all particles of the state are far apart, has a very simple form. In fact, considering the system in a box of length l and requiring periodic or antiperiodic boundary conditions for the asymptotic wave function, it is possible to obtain Bethe ansatz equations. The second part is based on statistical mechanics analyzes of the system; in the thermodynamic limit $l \to \infty$ - that means to pass from particles to particle densities and so for $l \to \infty$ we assume $N \to \infty$ and $\rho =$cost. - we determine the dominant microscopic configurations of the system consistent with a given set of macroscopic variables. The Bethe ansatz equations then lead to nonlinear integral equations named *TBA equations*.

Using the word “ansatz” for this method is somewhat misleading: as we shall see in detail below, the TBA just follows from the fact that the scattering is purely elastic; no additional assumption are requested. In particular, it is not necessary to know the Lagrangian formulation of the theory considered. The usual Bethe ansatz starts when one considers some Hamiltonian and has to prove that the ansatz provides a complete set of eigenstates.

It’ important to specify that the asymptotic wave function approximation can lead to exact results also in the case that the system has a nonzero density, i.e. the average distance between particles is finite. We are motivated to expect exact results from this approximation in the infinite-volume limit $l \to \infty$ because of the existence of a *virial expansion*[6] for thermodynamic quantities. Dashen, Ma and Bernstein [45] have shown that the nth term in this expansion is determined by the scattering matrix elements describing the scattering of n particles.
We shall see that TBA provides a “summed up version” of the virial expansion. This is why TBA should give exact results for any thermodynamic quantity of the system.

[6]Remember that a virial expansion is something like a generalization of a determined thermodynamic law. For instance, take the perfect gas law $\frac{\beta P}{\rho} = 1$, where $\beta = (kT)^{-1}$ and P is the pressure, ρ is the density and T is the temperature. The virial generalization or expansion takes the form $\frac{\beta P}{\rho} = 1 + \sum_{i=1}^{\infty} B_{i+1}(T)\rho^i$.

4.3.1 Thermodynamics of a two-dimensional relativistic purely elastic scattering theory.

We start - in euclidean formalism - with a flat (in the sense of its metric) torus generated by two orthogonal geodesic circles C and B of circumference R and L, respectively. We apply a cartesian coordinates system, where C is on the x-axis and B is on the y-axis, as it is already drawn in Fig.(4.1). There are two Hamiltonians that can be constructed for a theory on this geometry, topologically equivalent:

a) we choose the space for our field states the theory on circle C. We denote the corresponding Hilbert space as $\mathscr{C}$. So, the y-axis is the time axis and the Hamiltonian is

$$H_C = \frac{1}{2\pi} \oint_C dx T_{yy(=00)}, \tag{4.33}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the theory. The Momentum operator is

$$P_C = \frac{1}{2\pi} \oint_C dx T_{xy} \tag{4.34}$$

and it is quantized with eigenvalues equal to $2\pi n/R$ with $n \in \mathbb{Z}$.

b) we choose the space for our field states theory on circle B. We denote the corresponding Hilbert space as $\mathscr{B}$. So, the x-axis is the time axis and the Hamiltonian and the Momentum operator are

$$\begin{aligned} H_B &= \frac{1}{2\pi} \oint_B dy T_{xx(=00)}, \\ P_B &= -\frac{1}{2\pi} \oint_B dy T_{xy} \end{aligned} \tag{4.35}$$

We start from the configuration **a**) and we take the limit for $L \to \infty$ (infinite time), $L \gg R$. In this limit the partition function, from (4.24), becomes

$$Z(R, L) \sim \exp[-E_0(R)L], \tag{4.36}$$

because the ground-state is the state with less energy with respect to other states.

If we start from configuration **b**), we have

$$Z(R, L) = \exp Tr[-RH_B], \tag{4.37}$$

If we take, in this configuration, the limit $L \to \infty$ is like taking the *thermodynamic limit* of the system.From statistical mechanics one has

$$Z = \exp[-\beta F(R)] = \exp[-\beta L f(R)] \tag{4.38}$$

with F the free energy, $\beta = (KT)^{-1}$ and $f(R)$ the free energy per unit volume. On the other hand, (4.37) in the thermodynamic limit behaves like $LRf(R)$. Comparing this result with (4.38) one obtain that

$$\frac{1}{R} = T, \tag{4.39}$$

where T is the temperature of the system. In fact, if one uses, like us, the unit system with c[7] $= \hbar = 1$, then Boltzmann's constant is equal to 1 and $T = [1/length]$.

From (4.36) we find that

$$E_0(R) = Rf(R). \tag{4.40}$$

Linking together E and f is possible due to the relativistic equal footing of (euclidean) time and space.

Now we can start to build the TBA equations. We remain in the configuration **b**) and, to avoid useless complications, we consider for the moment a scattering theory of a single neutral particle of mass m and a pair scattering amplitude $S(\theta_1 - \theta_2)$. We recall that energy can be represented by $e_i(\theta) = m \cosh \theta_i$, momentum by $p_i(\theta) = m \sinh \theta_i$ and that S-matrix satisfies unitarity and crossing symmetry.

In the thermodynamic limit, the number of particles grows $\sim L$ and the spectrum of rapidities, obtained from (4.31), "condenses", i.e. the distance between two adjacent values behaves as $\theta_i - \theta_{i+1} \sim 1/mL$. For this reason, it is tempting now to move from a discretum to a continuum description, introducing the *rapidity density of particles* $\rho_1(\theta)$. Taking a small rapidity interval $\Delta\theta$ with n particles inside, it is possible to define

$$\rho_1(\theta) = n/\Delta\theta. \tag{4.41}$$

This definition is good if $(mL)^{-1} \ll \Delta\theta \ll 1$.

Bethe ansatz equations (4.31) in the continuum limit becomes[8]

$$mL \sinh \theta_i + \int d\theta' \delta(\theta_i - \theta')\rho_1(\theta') = 2\pi n_i. \tag{4.42}$$

[7] Speed of light.

[8] Here and below, the integrals are supposed to go from $-\infty$ to ∞ if not otherwise stated.

If we take a generic θ_i and a generic n_i, so if we take

$$mL\sinh\theta + \int d\theta'\delta(\theta-\theta')\rho_1(\theta') = 2\pi n, \tag{4.43}$$

with $n \in \mathbb{Z}$, we find the solution for *all possible* values of θ, or, in other words, we find the distribution of θ. The situation is analogous to that for a system of free particles, where the set of allowed levels is determined by the one-particle quantization condition and one talks about occupied and free levels, in this case ρ_1 and ρ. The only difference with the free case is that now the set of levels is organized self-consistently with the particle distribution.
Introducing the level density $\rho(\theta) = \partial n(\theta)/\partial\theta$ - with $\partial n(\theta)/\partial\theta \sim \Delta n/\Delta\theta$ and $\Delta n(\theta)$ the interval of solution of the $\Delta\theta$ interval - we obtain

$$2\pi\rho(\theta) = mL\cosh\theta + \int d\theta'\phi(\theta-\theta')\rho_1(\theta'), \tag{4.44}$$

where $\phi = \partial\delta/\partial\theta$.
The Hamiltonian now reads

$$H_B = \int d\theta m\cosh\theta\rho_1(\theta). \tag{4.45}$$

The number of different distributions in the interval $\Delta\theta_\alpha$, with $g_\alpha \sim \Delta\theta_\alpha\rho(\theta_\alpha)$ =number of levels and $n_\alpha \sim \Delta\theta_\alpha\rho_1(\theta_\alpha)$=number of particles in that level, is

$$\frac{g_\alpha!}{n_\alpha!(g_\alpha-n_\alpha)!} \tag{4.46}$$

for the fermionic case and

$$\frac{(g_\alpha+n_\alpha-1)!}{n_\alpha!(g_\alpha-1)!} \tag{4.47}$$

for the bosonic case.
We can define the entropy of the system to be $S(\rho,\rho_1) = \ln W(\rho,\rho_1)$, where W is the so-called Boltzmann's W. We can find that the entropy amounts to

$$S_{\text{Fermi}} = \int d\theta\,[\rho\ln\rho - \rho_1\ln\rho_1 - (\rho-\rho_1)\ln(\rho-\rho_1)] \tag{4.48}$$

in the fermionic case and to

$$S_{\text{Bose}} = \int d\theta\,[-\rho\ln\rho - \rho_1\ln\rho_1 + (\rho+\rho_1)\ln(\rho+\rho_1)] \tag{4.49}$$

in the bosonic case.

From statistical mechanics, we know that

$$\langle H \rangle = F + TS, \tag{4.50}$$

or, in our "language",

$$-RLf[\rho, \rho_1] = -RH_B[\rho_1] + S[\rho, \rho_1]. \tag{4.51}$$

We introduce the "pseudo-energy" $\epsilon(\theta)$ as

$$\begin{aligned} \frac{\rho_1}{\rho} &= \frac{e^{-\epsilon}}{1+e^{-\epsilon}} \qquad \text{fermionic case;} \\ \frac{\rho_1}{\rho} &= \frac{e^{-\epsilon}}{1-e^{-\epsilon}} \qquad \text{bosonic case.} \end{aligned} \tag{4.52}$$

We know that, for a system with constant volume and temperature, the free energy has a minimum in the equilibrium configuration. If we take the functional derivative of (4.51), that is

$$-R\frac{\delta H_B}{\delta \rho_1} + \frac{\delta S}{\delta \rho} + \frac{\delta S}{\delta \rho_1} = 0 \tag{4.53}$$

we find, using (4.44), the *extremum condition* for (4.51)

$$\begin{aligned} &-Rm\cosh\theta + \epsilon(\theta) + \int \frac{d\theta'}{2\pi}\phi(\theta-\theta')\ln(1+e^{-\epsilon(\theta')}) = 0 \qquad \text{fermionic case;} \\ &-Rm\cosh\theta + \epsilon(\theta) - \int \frac{d\theta'}{2\pi}\phi(\theta-\theta')\ln(1-e^{-\epsilon(\theta')}) = 0 \qquad \text{bosonic case.} \end{aligned} \tag{4.54}$$

With the help of (4.44), (4.45), (4.48) or (4.49), (4.51), (4.52) and (4.54) we can easily get

$$Rf(R) = \mp m \int \frac{d\theta}{2\pi}\cosh\theta \ln(1 \pm e^{-\epsilon(\theta)}), \tag{4.55}$$

with $+$ for bosonic case and $-$ for fermionic case.

In the general case the purely elastic scattering theory is described by a symmetric $M \times M$ matrix of two-particle transition amplitudes $S_{ab}(\theta)$, where $a, b = 1, \cdots, M$ label the particle types.
In the TBA approach one considers M level densities $\rho^{(a)}(\theta)$ and M particle densities $\rho_1^{(a)}(\theta)$. Eq. (4.55) turns out into a system of integral equations

$$\rho^{(a)}(\theta) = \frac{m_a L}{2\pi}\cosh\theta + \phi_{ab} * \rho_1^{(b)}(\theta) \tag{4.56}$$

with $\phi_{ab}(\theta)$[9] $= -i\frac{d}{d\theta}\ln S_{ab}(\theta)$ and $\phi * \rho_1(\theta) = \int \frac{d\theta'}{2\pi}\phi(\theta-\theta')\rho_1(\theta')$ is the convolution operator.

With the obvious definition

$$\rho_1^{(a)}(\theta) = \frac{e^{-\epsilon_a(\theta)}}{1 \pm e^{-\epsilon_a(\theta)}}\rho^{(a)}(\theta),$$

$$L_a(\theta) = \pm \ln(1 \pm e^{-\epsilon_a(\theta)}), \tag{4.57}$$

where upper signs correspond to fermionic and lower signs to bosonic type, we obtain the TBA equations in unified form

$$-m_a R\cosh\theta + \epsilon_a(\theta) + \sum_{b=1}^{M}\phi_{ab} * L_b(\theta) = 0. \tag{4.58}$$

If we take into account also the *chemical potential* μ, the Lagrangian multipliers related to the conservation of the particle number $N_{tot} = \sum_{b=1}^{M} N_b$, we can find that the TBA equations in unified form:

$$-m_a R\cosh\theta + \epsilon_a(\theta) + \mu_a R + \sum_{b=1}^{M}\phi_{ab} * L_b(\theta) = 0. \tag{4.59}$$

The extremal free energy per unit volume $f(R,\mu)$ is

$$Rf(R,\mu) = -\sum_{a=1}^{M} m_a \int \frac{d\theta}{2\pi} L_a(\theta,r,\mu)\cosh\theta + R\sum_{a=1}^{M}\mu_a\frac{N_a}{L} \tag{4.60}$$

From $f = -P + \sum_a \mu_a N_a/L$, where P is the pressure, we find that

$$P = T\sum_{a=1}^{M} m_a \int \frac{d\theta}{2\pi} L_a(\theta,r,\mu)\cosh\theta. \tag{4.61}$$

We note also the important identity

$$\rho^{(a)}(\theta) = \frac{L}{2\pi}\frac{\partial\epsilon_a(\theta)}{\partial R} \tag{4.62}$$

and we recall from (4.40) that the vacuum energy of the theory in the configuration **a**) is

$$E_0(R) = Rf(R,\mu=0) \tag{4.63}$$

[9] ϕ is called the matrix *kernel*

From (4.63) it is possible to find the finite-size ground-state scaling function

$$\tilde{c}(r) = \frac{3}{\pi^2} \sum_{a=1}^{M} \int d\theta L_a(\theta) \hat{m_a} r \cosh\theta, \tag{4.64}$$

where $\hat{m_a} = m_a/m_1$.

If we can resume this section in few words, thanks to the TBA equations (4.59), from the choice of T and μ we can find ϵ and then from these pseudo-energies we are able to find densities, pressure, entropy, ground-state energy, etc... For instance, from (4.61) we can calculate $dP = SdT + \sum_a \mu_a N_a/L$.

4.3.2 The UV limit.

In section 4.1.2 we have defined the UV and IR limit. We want to evaluate some properties in the UV limit. Often, in the literature, UV solutions are called *kink solutions* and they are denoted by the upper label ^{kink}.

Taking the derivative with respect to θ of (4.58), if $r \to 0$ ($R \to 0$), $\epsilon_a(\theta)$ is constant in the region $-|\ln(2/r)| \ll \theta \ll |\ln(2/r)|$. We call simply e_a this constant. Note that this "flat" region becomes bigger and bigger as r becomes smaller and smaller[10]. It is possible to find that

$$\begin{aligned} e_a &= \pm \sum_{b=1}^{M} N_{ab} \ln(1 \pm e^{-e_b}), \\ N_{ab} &= -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \phi_{ab}(\theta) = -\frac{1}{2\pi}(\chi_{ab}(\infty) - \chi_{ab}(-\infty)) \end{aligned} \tag{4.65}$$

From (4.58) it is possible to see that $\epsilon_a(\theta)$ and $L(\theta)$ are even functions. We can evaluate integral replacing the lower boundary $-\infty$ with 0 and multiply by 2. It is easy to note that the value of the region of integration where $\cosh\theta$ cannot be approximated by $\exp(\theta)/2$ is of order 1. For this reason we neglect this region of integration, due to its non influential contribution to the whole integration (in fact, for $r \to 0$ the integrand in this region goes to 0 too).

Now we can replace $\cosh\theta$ by $\frac{1}{2}e^{\theta}$, ϵ by $\overline{\epsilon}$, L by $\overline{L} = \pm\ln(1 \pm \exp(-\overline{\epsilon}))$. This quantities are defined by the following identity

$$\frac{1}{2} r \hat{m}_a e^{\theta} = \overline{\epsilon}_a(\theta) + \sum_{b=1}^{M} \phi_{ab} * \overline{L}_b(\theta). \tag{4.66}$$

[10] In the UV limit the masses of the particles go to zero, that means for this particles to "become" photons. They have rapidity equal to ∞ and the integral in (4.67) can be evaluated in this limit. This is the physical reason for our approximation.

$\overline{\epsilon}$ and $\overline{L}$ are constant for $\theta \ll |\ln(2/r)|$. Since $\overline{\epsilon}$ grows as the exponent of θ for $\theta \to \infty$, $\overline{L}$ decays as a double exponent in the same limit.
We have also

$$\tilde{c} = \tilde{c}_{\min}(0) = \frac{6}{\pi^2} \sum_{a=1}^{M} \lim_{r \to 0} \int_0^\infty \frac{d\theta}{2} \overline{L}_a(\theta) r \hat{m}_a e^\theta. \tag{4.67}$$

After integration, we arrive at the final result:

$$\tilde{c} = \sum_{b=1}^{M} \tilde{c}_\pm(e_a), \tag{4.68}$$

where

$$\tilde{c}_\pm(e) = \pm \frac{6}{\pi^2} \left[\left(\int_e^\infty dx \ln(1 \pm e^{-x}) \right) + \frac{1}{2} e \ln(1 \pm e^{-e}) \right] \tag{4.69}$$

$$\tilde{c}_\pm(e) = \frac{6}{\pi^2} \times \begin{cases} L\left(\frac{1}{1+e^e}\right) & \text{for bosonic type,} \\ L\left(e^{-e}\right) & \text{for fermionic type.} \end{cases} \tag{4.70}$$

We have introduced the function $L(x)$, called *Roger's dilogarithm*

$$L(x) = -\frac{1}{2} \int_0^x dy \left(\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y} \right). \tag{4.71}$$

It is worth remembering that $+$ is for fermionic type particles and $-$ if for bosonic type ones.

We see that we can associate a finite-size scaling coefficient $\tilde{c}_a$ to each particle species in the scattering theory, with the total $\tilde{c}$ given by their sum. We stress that

$$\begin{aligned} \tilde{c}_+(0) &= \frac{1}{2}; \\ \tilde{c}_-(0) &= 1; \end{aligned} \tag{4.72}$$

as in unitary CFT.

4.3.3 The IR limit.

In the $r \to \infty$ limit, we can suppose from (4.58) that

$$\epsilon_a(\theta) = r \hat{m}_a \cosh\theta + O(e^{-r}), \tag{4.73}$$

where the $O(e^{-r})$ correction is understood to possibly include powers of r. Put (4.73) in (4.55), with the help of (4.63) and (4.21) we find the first order approximation of the ground state finite-size scaling function

$$\begin{aligned} \tilde{c}_{\min}(r) &= \frac{6}{\pi^2}\sum_{a=1}^{M}\hat{m}_a\int_0^\infty d\theta\cosh\theta e^{-r\hat{m}_a\cosh\theta}\left(1+O(e^{-r})\right) = \\ &= \frac{6}{\pi^2}\sum_{a=1}^{M}\hat{m}_a K_1(\hat{m}_a r) + O(e^{-2r}). \end{aligned} \tag{4.74}$$

where K_1 is the modified Bessel function of order 1. It is possible to work out higher order terms too, that may be shown to contain interesting information on the scattering theory. For details see [46] and [49].

4.4 From CFT to QFT in $(1+1)$ dimension. Integrability conservation.

A conformal field theory is a theory without a mass-scale, that is, a massless theory. If we have a scale of mass, at least one fundamental particle whose mass is not zero, the theory is not conformal. We would be sorry about this, but, fortunately, there are some theories which are *perturbations* of conformal theories and often they don't break the integrability of the system. In all cases considered so far, there is a single perturbing parameter λ and the perturbed theory has only massive excitations (with the mass scale being a function of λ that goes to zero if $\lambda \to 0$). The relevant things are that the massive (perturbed) theory is described by a factorizable S-matrix and that these S-matrices are often related to Lie Algebras. This is the reason why these theories are usually referred to as *ADE scattering theories*, because only A, D and E series of simply-laced Lie algebras are in correspondence with these kind of theories.

We call a theory with an action

$$S = S_{CFT} + \lambda\int d^2x\Phi(x), \tag{4.75}$$

a "perturbation of a CFT", where $\Phi(x)$ is a relevant (i.e. with $\Delta < 1$) field of the CFT we are dealing with. Relevant means that it must take the theory out of the conformal point, giving a mass to the theory. Some fields aren't relevant (for instance, $\Phi_{1,1} \equiv \mathbb{I}$ or the fields with $\Delta > 1$) and don't give mass to the theory.

Some technical remarks are in order here. Lie Algebras can be divided in

two classes: Simple Lie Algebras (finite dimensional) and Affine Lie Algebras (infinite dimensional). Each of them is classified by nine "families", every family being characterized by different properties. Simple Lie Algebras are: $A_{r\geq 1}(su(r+1))$, $B_{r\geq 2}(so(2r+1))$, $C_{r\geq 3}(sp(2r))$, $D_{r\geq 4}(so(2r))$, E_8, E_7, E_6, F_4, G_2. Affine Lie Algebras (something called $\hat{\mathfrak{g}}$ or $\mathfrak{g}^{(1)}$) are obtained from Simple one ($\mathfrak{g}$) adding one more root to the set of simple roots of $\mathfrak{g}$. This root is $\alpha_0 \equiv (-\theta; 0; 1) = -\theta + \delta$ where θ is the highest root of $\mathfrak{g}$ and $\delta = (0; 0; 1)$. We do not consider here the wider class of "twisted" affine Lie Algebras, usually called $\mathfrak{g}^{(2)}$. For details see [52] or the complete mathematical monograph by V. Kac [53].
Every Simple (Affine) Lie Algebra have a characteristic list of *exponents*, that is the degree of Casimir operators[11] minus one and a natural number called the *Coxeter number h*.

Now, the spin of the conserved operators in the perturbed CFT are in one-to-one correspondence to the exponents of the related Lie Algebra, modulo its Coexter number. If in $(1+1)$ CFT we had an infinite series of conserved charges of all possible odd spins, now, in the perturbed CFT, we have a different but anyway infinite series of conserved charges with definite spin.
This is a first indication of a very relevant correspondence between Lie Algebras and S-matrix theory of massive *well* perturbed (out of the *conformal point*) theories.
Another indication is that the number of particles in the S-matrix theory related to an algebra $\mathfrak{g}$ is equal to the rank r of the algebra itself. In fact, we have a one-to-one correspondence between particles and *Dynkin diagram*[12]'s nodes. In Tab. 4.1 we sketch the Dynkin diagram for each Simple Lie Algebra. To be more explicit, we give the complete procedure used, in general, to relate a S-matrix of purely elastic massive (perturbed) theory with a conformal (unperturbed) theory:

1. start with r particles,
2. normalize m_1 (the lightest mass) to 1, then we find a "good" mass spectrum of the theory (related to a precise Lie Algebra);
3. write down the $S_{11}(\theta)$ element, where 1 labels the lightest particle;
4. find, from bootstrap, the others elements of the S-matrix S_{ab};
5. look for the corresponding UV conformal theory by using TBA approach.

[11] These operators are the only ones that commute with the generators of the Lie group associated with its algebra.

[12] They are very useful graphic representation of the Simple (Affine) Lie Algebras.

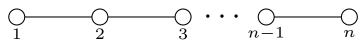
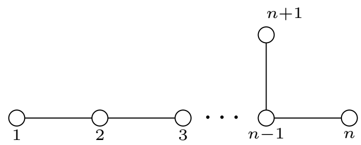
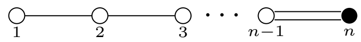
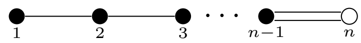
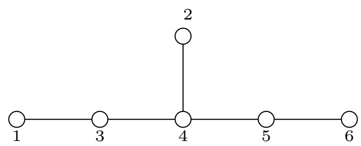
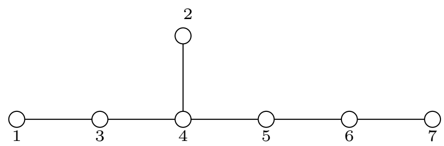
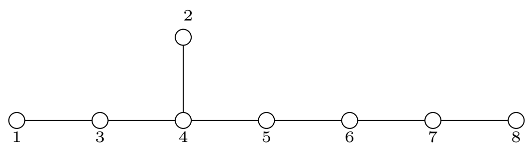
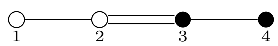
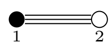
Algebra	*Dynkin diagram*	h	*Dim.*	*Exponents*
A_n		$n+1$	n^2+2n	$1, 2, \cdots, n$
D_n		$2n-2$	$2n^2-2$	$1, 3, \cdots, 2n-3,\ n-1$
B_n		$2n-1$	$2n^2+n$	$1, 3, \cdots, 2n-1$
C_n		$n+1$	$2n^2+n$	$1, 3, \cdots, 2n-1$
E_6		12	78	$1, 4, 5, 7, 8, 11$
E_7		18	133	$1, 5, 7, 9, 11, 13, 17$
E_8		30	248	$1, 7, 11, 13, 17, 19, 23, 29$
F_4		9	52	$1, 5, 7, 11$
G_2		4	14	$1, 5$

Table 4.1: Simple Lie Algebras. Black nodes refer to short roots, white ones to long roots. *Dim.* means *dimension.*

4.4.1 ADE for purely elastic S-matrices.

We follow the Dorey approach to the ADE purely elastic S-matrices, [54]. A $(1+1)$ dimensional purely elastic scattering theory has a factorizable and diagonal S-matrix (see sections 3.2.4 and 3.2.5). In the so called ADE scattering theories the poles (also called *fusing angles*) u_{ab}^{c} are all integer multiples of π/h, h being the Coxeter number of $\mathfrak{g}$, the Lie algebra of rank r associated to the theory.
We have already said that non-trivial solutions occur if the spin s modulo h of the conserved charges is equal to an exponent of $\mathfrak{g}$.

Furthermore, each of the r particles in the theory may be assigned to a node on the Dynkin diagram of $\mathfrak{g}$, in such a way that the set of conserved charges of spin s can be put together in a vector

$$\gamma_s = (\gamma_s^{a_1}, \cdots, \gamma_s^{a_r}), \tag{4.76}$$

that turns out to be the eigenvector of the *incidence*[13] matrix G of the $\mathfrak{g}$ Dynkin diagram, with eigenvalue $g_s = 2\cos(\pi s/h_{\mathfrak{g}})$. If we set $s=1$ we find from $G\gamma_s = g_s\gamma_s$ the masses of the particles in the theory.

If we refer to the usual Cartan-Weyl construction of Lie Algebras and we call (Ξ) the the set of roots of $\mathfrak{g}$ and $\Pi = (\alpha_1, \cdots, \alpha_r)$ a set of simple roots of $\mathfrak{g}$, we can find that Π is the union of two sets of orthogonal simple roots

$$\Pi = (\alpha_1, \cdots, \alpha_k) \cup (\beta_1, \cdots, \beta_{r-k}), \qquad (\alpha_i, \alpha_j) = (\beta_i, \beta_j) = \delta_{ij} \tag{4.77}$$

The *Weyl group* is the group of all reflections

$$w_\alpha(x) = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}, \qquad \alpha \in \Xi. \tag{4.78}$$

The *Coxeter elements* of the Weyl group are elements of the form $w_{\alpha_1} \cdots w_{\alpha_r}$, with $\alpha_i \in \Pi$. The Coexter element of (4.77) is $w = w_{\alpha_1} \cdots w_{\alpha_k} w_{\beta_1} \cdots w_{\beta_{r-k}}$. It has a period equal to the Coexter number h, so the group generated by w is isomorphic to Z_h.

If we define

$$a_s = \sum_i \gamma_s^{\alpha_i} \hat{\alpha}_i, \qquad b_s = \sum_i \gamma_s^{\beta_i} \hat{\beta}_i \tag{4.79}$$

where $(\hat{\alpha}, \hat{\beta})$ is the dual of the simple root basis (α, β), then

$$(\alpha_i, a_s) = \gamma_s^{\alpha_i}, \qquad (\beta_i, a_s) = 0, \qquad (\alpha_i, b_s) = 0, \qquad (\beta_i, b_s) = \gamma_s^{\beta_i}. \tag{4.80}$$

[13]The elements of this matrix are integer numbers. G_{ab} is equal to the number of links connecting nodes b and a.

We define two projectors P_s into the sub-space spanned by a_s and b_s

$$P_s(\alpha_i) = \gamma_s^{\alpha_i}\hat{a_s} \qquad P_s{-}\beta_j = \gamma_s^{\beta_j}\hat{b_s}, \tag{4.81}$$

where $(\hat{a_s}, -\hat{b_s})$ are the dual to (a_s, b_s) in that subspace.

The Coexter element w represents a rotation by $2\pi s/h$ in each subspace spanned by α or β. We have

$$\begin{aligned} P_s(w^p\alpha_i) &= \gamma_s^{\alpha_i} e^{i(2p+1)\pi s/h}, \\ P_s(w^p(-\beta_j)) &= \gamma_s^{\beta_j} e^{i(2p)\pi s/h}. \end{aligned} \tag{4.82}$$

We can now give a general expression for S_{ab} in ADE scattering theories

a) If the particles a and b are of the type α[14]

$$S_{ab} = \prod_{p=0}^{h-1} \{2p+1\}_+^{(\alpha_a, w^p\alpha_b)}. \tag{4.83}$$

b) If the particle a is of the type α and b of the type β[15]

$$S_{ab} = \prod_{p=0}^{h-1} \{2p\}_+^{(\alpha_a, -w^p\beta_b)}. \tag{4.84}$$

c) If the particles a and b are of the type β

$$S_{ab} = \prod_{p=0}^{h-1} \{2p-1\}_+^{(\beta_a, -w^p\beta_b)}. \tag{4.85}$$

where we have used the notation

$$\begin{aligned} \{x\}_+ &= (x-1)^+(x+1)^+, \\ (x)^+ &= \sinh\left(\frac{\theta}{2} + \frac{i\pi x}{2h}\right). \end{aligned} \tag{4.86}$$

With this formalism it is possible to find that ([56]) for $\theta \neq 0$

$$S_{ab}\left(\theta + \frac{i\pi}{h}\right) S_{ab}\left(\theta - \frac{i\pi}{h}\right) = \prod_c S_{ac}(\theta)^{G_{ab}}, \tag{4.87}$$

with G_{ab} being the incidence matrix, defined above.

We give here a list of ADE scattering theories to which correspond particular CFTs perturbed by an operator; normalizing the mass of the fundamental particle to unity, we write down the spectrum of the other masses:

[14] in the Dynkin diagram at each node corresponds a simple root

[15] Remember that $S_{ab} = S_{ba}$

1. the A_n series is identified with the CFT's of Z_{n+1} parafermions with central charge $c(A_n) = 2n/(n+3)$, perturbed by a primary field of dimension $\Delta(A_n) = 2/(n+3)$. The mass spectrum of the theory is $m_a = \sin\left(\frac{\pi a}{n+1}\right) / \sin\left(\frac{\pi}{n+1}\right)$, with $a = 1, \cdots, n$.

2. The D_n series S-matrices are that of the sine-Gordon scattering theory at the reflectionless points $\beta^2 = 8\pi/n$. Therefore the central charge here is independent of n : $c(D_n) = 1$ and the perturbation is of dimension $\Delta(D_n) = 1/n$. We have a different mass spectrum depending if n is odd or even.

3. The scattering theory related to E_6 was supposed to be a perturbation of the minimal model $\mathscr{M}(\frac{6}{7})$ (tricritical three-state Potts model) with $c(E_6) = 6/7$, perturbed by the operator $\phi_{(1,2)}$ of dimension $\Delta(E_6) = 1/7$. We have 6 particles, $1, \overline{1}, 2, 3, \overline{3}, 4$ which masses are: $m_1 = m_{\overline{1}} = 1$, $m_2 = \sqrt{2}$, $m_3 = m_{\overline{3}} = (1+\sqrt{3})/\sqrt{2}$, $m_4 = 1 + \sqrt{3}$.

4. The E_7-related S-matrix corresponds to the perturbation of $\mathscr{M}(\frac{4}{5})$ (tricritical Ising model) with $c(E_7) = 7/10$ by a primary field $\phi_{(1,2)}$ of dimension $\Delta(E_7) = 1/10$. This model has 7 different particles, with the following mass spectrum: $m_1 = 1$, $m_2 = 2\cos(5/18\pi)$, $m_3 = 2\cos(1/9\pi)$, $m_4 = 2\cos(1/18\pi)$, $m_5 = 4\cos(1/18\pi)\cos(5/18\pi)$, $m_6 = 4\cos(1/9\pi)\times$ $\times\cos(2/9\pi)$, $m_7 = 4\cos(1/18\pi)\cos(1/9\pi)$.

5. The E_8 reflectionless scattering theory, which corresponds to the magnetic perturbation (dimension $\Delta(E_8) = 1/16$) of the critical Ising model ($\mathscr{M}(\frac{3}{4})$, $c(E_8) = 1/2$). We write down its mass spectrum: $m_1 = 1$, $m_2 = 2\cos(1/5\pi)$, $m_3 = 2\cos(1/30\pi)$, $m_4 = 4\cos(1/5\pi)\times$ $\times\cos(7/30\pi)$, $m_5 = 4\cos(1/5\pi)\cos(2/15\pi)$, $m_6 = 4\cos(1/5\pi)\times$ $\times\cos(1/30\pi), m_7 = 8\cos^2(1/5\pi)\cos(7/30\pi), m_8 = 8\cos^2(1/5\pi)\cos(2/15\pi)$.

4.4.2 Y-system for purely elastic scattering.

The TBA equations for ADE scattering theories, thanks to (4.87), become more useful and user-friendly in order to evaluate the finite-size scaling function (4.64). They assume a very simple form, called *universal* where the kernel is always the same function multiplied by the incidence matrix G_{ab} of the corresponding Lie Algebra.

For all ADE scattering theories enumerated at the end of the previous section, (4.59) can be written in the so called *universal form*

$$\begin{cases} -\nu_a + \epsilon_a + \dfrac{1}{\pi}\displaystyle\sum_b G_{ab}\phi_h * \{\nu_b - \ln\left[1+\exp(\epsilon_b)\right]\} = 0, \\ \nu_a = Rm_a \cosh\theta. \end{cases} \tag{4.88}$$

We have defined the *universal kernel*

$$\phi_h(\theta) = \frac{h}{2\cosh\frac{h\theta}{2}}. \tag{4.89}$$

Thanks to the Fourier transform identity

$$\tilde{\phi}_{ab}(k) = \int_{-\infty}^{\infty} \phi_{ab}(\theta)\exp(ik\theta)d\theta \tag{4.90}$$

for $\phi_{ab}(\theta) = -i\frac{d}{d\theta}\log S_{ab}(\theta)$, it is possible to obtain ([50]) the following matrix relation

$$\left(\delta_{ab} - \frac{1}{2\pi}\phi_{ab}(k)\right)^{-1} = \delta_{ab} - \frac{1}{1\cosh(k/h)}G_{ab}, \tag{4.91}$$

on which is based the equivalence between (4.58) and (4.88). The equation (4.88) is called universal because it is equal for every system a part from the adjacency matrix.
Note that everything depends only on the Coexter number h. The masses of the particles are known to be encoded on the *Perron-Frobenius*[16] eigenvector Ψ_G of G: $m_a = m_1\Psi_G^a$. If we define

$$y_a(\theta) \equiv \exp(\epsilon_a(\theta)), \tag{4.92}$$

we can check that (4.88)-solutions are also solutions for the *Y-system* ([56])

$$y_a(\theta + i\pi/h)y_a(\theta - i\pi/h) = \prod_b \left[1 + y_b(\theta)\right]^{G_{ab}}. \tag{4.93}$$

Taking the logarithmic form of (4.87)

$$\log S_{ab}\left(\theta + \frac{i\pi}{h}\right) + \log S_{ab}\left(\theta - \frac{i\pi}{h}\right) = \sum_c G_{bc}\log S_{ac}(\theta) - 2i\pi\Theta(\theta)G_{ab}, \tag{4.94}$$

[16]Is the eigenvector with the greater eigenvalue of the G_{ab} matrix. It is equal to γ_s with $s = 1$ in (4.76).

where the term proportional to the step function

$$\Theta = \lim_{\epsilon \to 0} \left[\frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{\epsilon} \right] = \begin{cases} 0 & \text{if} \quad x < 0; \\ \dfrac{1}{2} & \text{if} \quad x = 0; \\ 1 & \text{if} \quad x > 0; \end{cases} \tag{4.95}$$

has to be introduced to take into account the correct prescription for the multivalued function $\log x$. If we want to include the point $\theta = 0$, according to (4.95) we must correct (4.87) as follows

$$S_{ab} \left(\theta + \frac{i\pi}{h} \right) S_{ab} \left(\theta - \frac{i\pi}{h} \right) = \prod_c S_{ac}(\theta)^{G_{bc}} \exp(-2i\pi G_{ab} \Theta(\theta)). \tag{4.96}$$

Taking the derivative of (4.94), with the usual form for ϕ_{ab}, and Fourier transform this equation we obtain

$$2 \cos \left(\frac{k\pi}{h} \right) \tilde{\phi}_{ab}(k) = \sum_c G_{bc} \tilde{\phi}_{ac}(k) - 2\pi G_{ab} \tag{4.97}$$

or

$$\tilde{\phi}_{ab}(k) = -2\pi \left[G \left(2 \cos \left(\frac{k\pi}{h} \right) - G \right)^{-1} \right]_{ab} . \tag{4.98}$$

Note that (4.98) is equivalent to (4.91), but is more useful. Fourier transform (4.58), then multiply both sides by $\delta_{ab} - \tilde{R}(k) G_{ab}$, where $\tilde{R}(k) = (2 \cosh(k\pi/h))^{-1}$, and finally use (4.97), we obtain the universal form of TBA equations (4.88).

The y_a functions can be shown to have a very interesting periodicity property (appeared for the first time in [50] and completely derived in [59])

$$y_a \left(\theta + i\pi P \right) = y_a(\theta), \tag{4.99}$$

for the D_n and E series, and

$$y_a \left(\theta + i\pi P \right) = y_{n-a+1}(\theta) \tag{4.100}$$

for the A_n series, where

$$P \equiv \frac{h+2}{h}. \tag{4.101}$$

In the UV limit we have found that the finite-size scaling function depends only by the constant value that $\epsilon_a(\theta)$ takes on the very large region $[0, \ln(2/r)]$, i.e. $\tilde{\epsilon}_a$. With the formalism developed in this section, we rewrite previous results ((4.68) and (4.69))

$$y_a \equiv e^{\tilde{\epsilon}_a}, \qquad N_{ab} = -\int_{-\infty}^{+\infty} \frac{d\theta}{2\pi}\phi_{ab}(\theta), \qquad y_a = \prod_b \left[1 + 1/y_b\right]^{N_{ab}}, \quad (4.102)$$

$$\tilde{c}_{\pm}(e) = \frac{6}{\pi^2} \times \begin{cases} L\left(\frac{1}{1+y}\right) & \text{bosonic type,} \\ L\left(\frac{1}{y}\right) & \text{fermionic type.} \end{cases} \quad (4.103)$$

For ADE scattering theories (4.102) can be rewritten in this form

$$y_a^2 = \prod_b (1 + y_b)^{G_{ab}}. \quad (4.104)$$

The solution of this system is the right value for the constant $\tilde{\epsilon}_a$, useful to evaluate ADE's $\tilde{c}$. Note also that the y_a are the stationary solutions of (4.93).

It can be shown that the dimension Δ of the perturbative operator Φ in (4.75) is

$$\Delta = 1 - \frac{1}{P}, \quad (4.105)$$

This is the only link between the perturbative operator and the related Lie Algebra corresponding to the Φ-perturbed massive S-matrix theory.

We have just discovered that, from the Y-system (4.93), we can deduce both the finite-size scaling function and the perturbation made to obtain the massive theory.

4.4.3 ADE for general S-matrices.

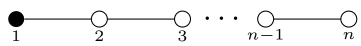
Many important theories (i.e. Sine-Gordon models (SG), $O(3)$ NLSMs, the Sausage model, etc...) have a factorized non-diagonal 2-particle S-matrix, which is the expression of a non-zero absorption and reflection amplitudes. Everything said so far holds for purely-elastic scattering theory and it can't work for general scattering theory. However, with some modification, we can use the TBA and the Bootstrap approaches also for these theories.
In general scattering theories, particles acquire *colors*, which can be represented by mass-less fictitious particles called *magnons*, which are embodied in the TBA equations. For the non-elastic scattering Bootstrap approach we follow the Karowsky method in [60]. Each particle can have a tower of magnons and each tower can be related in some cases to a Dynkin diagram. We could have a Dynkin diagram for the massive part of the spectrum too. Graphically, we draw the massive diagram with black nodes and from each

massive node it starts a magnon diagram.
It exists a general formula for the TBA of these theories and, in some cases, it is possible to reduce TBA equations in a universal form. In these cases, only ADE adjacency matrices are permitted and it is possible to associate to these general scattering theories a particular product of many Dynkin diagram. This product is labelled by a new operation ([56])

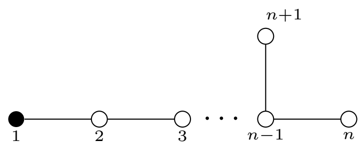
$$ADE \diamond ADE \tag{4.106}$$

For instance, the TBA for magnons, for each massive node, in the ADE case can be written as $A_1 \diamond ADE$, like sketched in Tab.(4.2). On the contrary, the ADE purely elastic scattering case as a TBA that, with the formalism just explained, is $ADE \diamond A_1$. In fact, we have a completely massive Dynkin diagram without colors. An example is shown in Tab.(4.3)

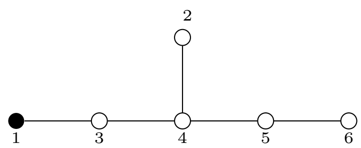
$A_1 \diamond A_n$



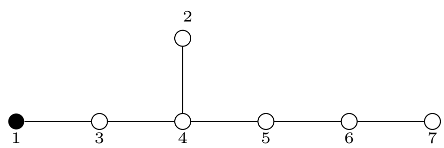
$A_1 \diamond D_n$



$A_1 \diamond E_6$



$A_1 \diamond E_7$



$A_1 \diamond E_8$

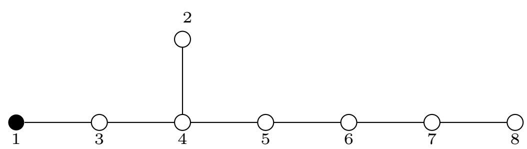


Table 4.2: $A_1 \diamond ADE$ scattering theories in their diagrammatic form. Black nodes represent the massive part of the particle.

$A_n \diamond A_1$

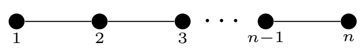


Table 4.3: $ADE\diamond A_1$ scattering theories in their diagrammatic form. Massive particles of the spectrum are in one-to-one correspondence with Dynkin diagram nodes.

The *magnonic* TBA $A_1\diamond ADE$ was proposed for the first time by Al. B. Zamolodchikov [55]. It has the general diagrammatic form

$$\begin{cases} \nu_a(\theta) = \epsilon_a(\theta) + \dfrac{1}{2\pi}\sum_b G_{ab}\,(\phi * L_b)\,(\theta), & \phi = \dfrac{1}{\cosh\theta}, \\ \nu_a = \begin{cases} \delta_a^k mR\cosh\theta & \text{for massive case, } \lambda < 0, \\ \frac{mR}{2}(\delta_a^k e^{\theta} + \delta_a^l e^{-\theta}) & \text{for massless case, } \lambda > 0. \end{cases} \end{cases} \tag{4.107}$$

The two nodes of the massless case are called *right-mover* and *left-mover*. In the massive case, k is the label of the massive node.
In the following sections we study in detail the TBA for the Sine-Gordon theory, a non elastic scattering theory with two massive particles (called soliton and anti-soliton). We shall see how, from the block-diagonal S-matrix, it will be possible to obtain the TBA equations and the corresponding universal form.

4.5 Integrability in quantum mechanics.

TBA techniques described in section 4.3 can be applied to purely-elastic scattering theories. In fact, the relativistic Bethe Ansatz already described in previous sections is valid only for those theories.
It is possible to formulate a Bethe Ansatz also for non-elastic scattering theories, but we need first new mathematical and physical tools, based on the integrability of some systems called *spin chain* models.
In the following, we introduce some basic notions on integrability of spin-chains, in order to derive the *Algebraic Bethe Ansatz*, which will be used for the derivation of the TBA equations for two non-elastic scattering theories, the SG and the SSM models. For the interested reader, we refer to ([67], [68]).

4.5.1 Spin chains, R-matrix and transfer matrix.

In order to study the ferromagnetic and anti-ferromagnetic behavior, Heisenberg proposed a one dimensional object, called spin-chain, where particles of spin j are sitting on N $(n = 1, \cdots, N)$ sites of a lattice and interact locally with the near-neighbor particles sitting at $n-1$ and $n+1$, with some local piece of Hamiltonian $H_{n,n+1}$, the total Hamiltonian being

$$H = \sum_{n=1}^{N-1} H_{n,n+1}. \tag{4.108}$$

At site n, the particle state is represented by a vector in a finite Hilbert space $V_n = \mathbb{C}^{2j+1}$. Periodic boundary conditions may be imposed. If we identify the first with the last site, we speak about *periodic chains*.

The Hamiltonian (4.108) is written in terms of spin operators S_n^k, where $k = x, y, z$. $n = 1, \cdots, N$ label runs on the sites of the chain. They are operators acting on the Hilbert space

$$\mathscr{H} = V_1 \otimes V_2 \otimes \cdots \otimes V_N \tag{4.109}$$

as identities on all V_m's with $m \neq n$ and, as usual, spin s representation operators on the site n

$$S_n^k = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \underbrace{S^k}_{n-\text{th position}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}, \tag{4.110}$$

where

$$\left[S^k, S^l\right] = i\epsilon^{klw} S^w. \tag{4.111}$$

In the case of $s = 1/2$, where $S^2 = \sum_i S^i S^i$ has eigenvalue $s(s+1)$, $S_k = \sigma_k/2$ and the spin chain Hamiltonian is

$$H = \sum_{n=1}^{N-1} \left(J^x \sigma_n^x \sigma_{n+1}^x + J^y \sigma_n^y \sigma_{n+1}^y + J^z \sigma_n^z \sigma_{n+1}^z\right) \tag{4.112}$$

with coefficients J^k. The original problem of Heisenberg assumed $J^x = J^y = J^z \equiv J$, so that

$$H = J \sum_{n=1}^{N-1} \left(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\right). \tag{4.113}$$

Here $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

Due to the symmetry in the 3 spin directions, this is called the XXX-chain, while the general case (4.112) is called XYZ-chain. A very important case, that finds a lot of applications, e.g. in condensed matter physics, is the XXZ-chain

$$H = J \sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z\right), \tag{4.114}$$

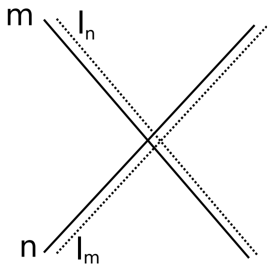


Figure 4.2: $R_{nm}^{(l_n,l_m)}$ graphic representation. Dotted line represents $l_{n(m)}$, solid line represents $n(m)$.

where $J^x = J^y = J$ and $\Delta = J^z/J$ is called *anisotropy*. In the case $s = 1$, the analog of the XXX-chain is given by the Hamiltonian

$$H = J \sum_{n=1}^{N-1} \left[\vec{S}_n \cdot \vec{S}_{n+1} - \left(\vec{S}_n \cdot \vec{S}_{n+1} \right)^2 \right], \tag{4.115}$$

where S_n^i are 3×3 matrices realizing the spin 1 representation of $SU(2)$. More general hamiltonians can be given for the XXZ and XYZ cases and/or for higher spin. Many of these Hamiltonians turn out to be integrable.
To make contact with integrable formalism, let us first introduce the so-called R-matrix as a family operators $R_{nm}^{(l_n,l_m)}(\lambda)$, depending on a parameter λ and acting on the tensor product of two local spaces $V_n \otimes V_m$, with $V_n \simeq \mathbb{C}^{l_n+1}$ and $l_n = 2s_n$. Loosely speaking, it means that the site n has a spin equal to s_n. Formally, n labels the site represented in the Hilbert space by the s_n-representation with respect to the group of symmetry which, acting on the site, leaves the Hamiltonian invariant.
When considered on the product of three spaces $V_n \otimes V_{\otimes} V_p$, $R_{nm}^{(l_n,l_m)}$ satisfies the *Yang-Baxter* (YB) equation

$$R_{nm}^{(l_n,l_m)}(\lambda_{nm}) R_{np}^{(l_n,l_p)}(\lambda_{np}) R_{mp}^{(l_m,l_p)}(\lambda_{mp}) = R_{mp}^{(l_m,l_p)}(\lambda_{mp}) R_{np}^{(l_n,l_p)}(\lambda_{np}) R_{nm}^{(l_n,l_m)}(\lambda_{nm}), \tag{4.116}$$

with $\lambda_{ij} \equiv \lambda_i - \lambda_j$. Graphically, we represent $R_{nm}^{(l_n,l_m)}$ by Fig.(4.2) and the YB equation becomes Fig.(4.3). Algebras underlying the YB equation at quantum level are a *deformation* of Kac-Moody algebras and are known as *quantum algebras*.

For instance, a solution of the YB equation in the $1/2 - XXX$ case is

$$R_{nm}^{(1,1)}(\lambda) = \lambda \mathbb{I} + i \mathscr{P}_{nm}^{(1,1)}, \tag{4.117}$$

where $\mathscr{P}_{nm}^{(1,1)}$ is the permutation operator exchanging V_n and V_m. It is possible

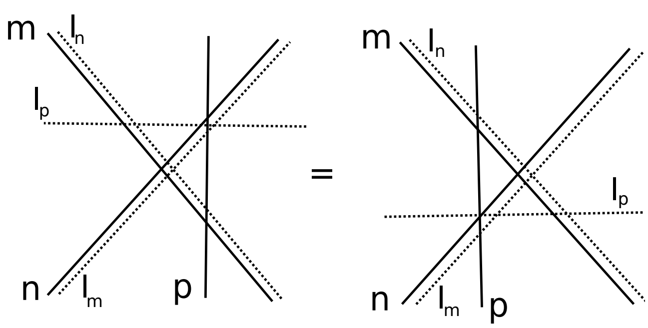


Figure 4.3: A schematic view of YB equation (4.116).

to prove that from the R matrix we can obtain the Hamiltonian of the system

$$\frac{d}{d\lambda}\check{R}^{XXX}_{12}\Big|_{\lambda=0} = \mathscr{P} \propto H^{XXX}_{12}, \tag{4.118}$$

where H^{XXX}_{12} is the two-sites XXX Hamiltonian and $\check{R} = \mathscr{P}R$. In fact, the permutation operator can be written as a function of the generators of the fundamental representation of the theory

$$\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\sigma^z + 1) & \sigma^- \\ \sigma^+ & \frac{1}{2}(-\sigma^z + 1) \end{pmatrix}, \tag{4.119}$$

with $\sigma^{\pm} = (\sigma^y \pm i\sigma^z)/2$. In order to generalize the study of the spin chain to the generic representation s_n, we define the *quantum Lax operator* L as

$$L^{(l_n,l_m)}_{nm}(\lambda) = \lambda\mathbb{I} + i\mathbb{P}^{(l_n,l_m)}_{nm}, \tag{4.120}$$

where

$$\mathbb{P}^{(l_n,l_m)}_{nm} = \begin{pmatrix} S^z_n + 1/2 & S^-_n \\ S^+_n & -S^z_n + 1/2 \end{pmatrix}. \tag{4.121}$$

S^i_n are generic elements of the s_n-representation of the group of symmetry. L satisfies the following fundamental algebraic relation with the R-matrix

$$R^{(l_n,l_m)}_{nm}(\lambda_{nm})L^{(l_n,l_p)}_{np}(\lambda_{np})L^{(l_m,l_p)}_{mp}(\lambda_{mp}) = L^{(l_m,l_p)}_{mp}(\lambda_{mp})L^{(l_n,l_p)}_{np}(\lambda_{np})R^{(l_n,l_m)}_{nm}(\lambda_{nm}). \tag{4.122}$$

In general, $L \in \mathrm{End}(A \otimes V)$, where A is the algebra defined by (4.122) and is often called the *auxiliary space*. Different choices of R matrix lead naturally to distinct algebras and distinct models.

For another example, we now consider the $1/2-XXZ$-chain. The two-site R-matrix for this model is

$$R=\begin{pmatrix}\sinh(i\mu-\lambda) & 0 & 0 & 0\\ 0 & \sinh\lambda & \sinh i\mu & 0\\ 0 & \sinh i\mu & \sinh\lambda & 0\\ 0 & 0 & 0 & \sinh(i\mu-\lambda)\end{pmatrix}, \tag{4.123}$$

with $\mu = 2k\pi/p$, $k,\ p\in\mathbb{N}$ and $k,\ p\neq 0$. It is possible to write R also as

$$R^{(1,1)}(\lambda)=\sinh\left(\lambda+i\mu\frac{1+\sigma^3\otimes\sigma^3}{2}\right)+i\sin\mu\left(\sigma^+\otimes\sigma^-+\sigma^-\otimes\sigma^+\right). \tag{4.124}$$

We check from (4.122) that for the $1/2-XXZ$ chain

$$\begin{aligned}&[S^+,\ S^-]=\frac{q^{2S^z}-q^{-2S^z}}{q-q^{-1}},\\ &[S^z,S^\pm]=\pm S^\pm,\end{aligned} \tag{4.125}$$

where q is a real number. Relations in (4.125) are the same relations between generators of the *quantum algebra* $U_q(SU(2))$. For this reason we say that the $1/2-XXZ$-chain is symmetric under $U_q(SU(2))$ transformations[17].

From the R matrix it is possible to reconstruct the Hamiltonian of the system. To see that, first of all we have to define the *monodromy matrix* $T^{(l)}(\lambda)$. Focusing on the case of a s_m-spin chain, the monodromy matrix is

$$T^{(l_a)}(\lambda)=\Pi_{(l_a)}\left(L^{(l_a,l_m)}_{aN}(\lambda)L^{(l_a,l_m)}_{aN-1}(\lambda)\cdots L^{(l_a,l_m)}_{a1}(\lambda)\right), \tag{4.126}$$

where $\Pi_{(l_a)}$ is the projection on the s_a-th representation, which is not the representation of the physical sites but a generic representation among all the representations of the group under which the Hamiltonian of the theory is invariant. For this reason we call the space of this representation the *auxiliary space*, labelled by a. One sees that $T^{(l_a)}\in \mathrm{End}(V_a\otimes\bigotimes_{i=1}^N V_{m,i})$, where V_a is the auxiliary space and $V_{m,i}$ are the Hilbert spaces of the s_m-spin chain sites. The monodromy and the R matrices are related together by the *fundamental algebraic relation* (FRT)

$$R^{(l_n,l_m)}_{nm}(\lambda_{nm})T^{(l_n)}(\lambda_n)T^{(l_m)}(\lambda_m)=T^{(l_m}(\lambda_m)T^{(l_n)}(\lambda_n)R^{(l_n,l_m)}_{nm}(\lambda_{nm}). \tag{4.127}$$

[17]The notation $U_q(SU(2))$ means "q-deformed universal covering of the algebra of special linear 2×2 matrices. Universal covering of an algebra $\mathfrak{g}$ is the algebra of all polynomials in the generators of $\mathfrak{g}$. "Special" here means "with determinant equal to 1".

Tracing over the auxiliary space we find the so-called *transfer matrix* $t_{l_a}^{(l_m)}(\lambda)$

$$t_{l_a}^{(l_m)}(\lambda) = Tr_A[T^{(l_a)}(\lambda)]. \tag{4.128}$$

It is possible to find that the transfer matrix forms a λ-family of commuting operators, i.e.

$$[t_{l_a}^{(l_m)}(\lambda),\ t_{l_b}^{(l_m)}(\lambda')] = 0. \tag{4.129}$$

The commutation relation (4.129) implies that factors of formal series expansion commute with each other. From $\log t(\lambda) = \sum_n \lambda_n I^{(n)}$, where $t(\lambda) \equiv t_1^{(1)}(\lambda)$, we can check from (4.129) that

$$[I^{(n)}, I^{(m)}] = 0 \qquad \forall n, m. \tag{4.130}$$

The elements $I^{(n)}$ are the so called *charges in involution.* The first two charges are the total momentum and the total energy. In fact, it is possible to show that

1.
$$\frac{d}{d\lambda}\log(t(\lambda))_{|\lambda=0} \propto H, \tag{4.131}$$
where H is the Hamiltonian of the system;

2.
$$t(0) = \Pi = \exp(-iP), \tag{4.132}$$
where P is the total momentum.

As it was conjectured, the $O(3)$ NLSM describes the continuum limit of the XXX-chain model with large integer (half-integer) spin. We think that the sausage model is related to the higher-spin XXZ model. However, the relation between microscopic parameters of a particular XXZ chain near criticality and our universality parameters (λ and m) remains to be established.

4.5.2 Algebraic Bethe Ansatz for $U_q(SU(2))$.

Suppose that we have a system with a $U_q(SU(2))$ symmetry. The appropriate form for the L-matrix is

$$L = \begin{pmatrix} e^{\lambda}A - e^{-\lambda}D & (q-q^{-1})B \\ (q-q^{-1})C & e^{\lambda}D - e^{-\lambda}A \end{pmatrix} \tag{4.133}$$

with

$$\begin{cases} A = D^{-1} \ \propto \ q^{S^z}; \\ B \ \propto \ S^-; \\ C \ \propto \ S^+. \end{cases} \tag{4.134}$$

A, B, C and D belong to some representation of $U_q(SU(2))$. Moreover, if we introduce the *Heisenberg-Weyl* group elements $\mathbb{X}$ and $\mathbb{Y}$ so that

$$\mathbb{X}\mathbb{Y} = q\mathbb{Y}\mathbb{X}, \tag{4.135}$$

it turns out that

$$\begin{cases} A = D^{-1} = \mathbb{X}, \\ B = \dfrac{1}{(q-q^{-1})}(q^{-s}\mathbb{X}^{-1} - q^s\mathbb{X})Y^{-1}, \\ C = \dfrac{1}{(q-q^{-1})}(q^{-s}\mathbb{X} - q^s\mathbb{X}^{-1})Y. \end{cases} \tag{4.136}$$

From (4.135) it's easy to see the q-deformation by the introduction of the quantum algebra. If $q \to 1$ we return to classical groups and algebras.

Now we can start our project to find the complete spectrum of a model with a $U_q(SU(2))$ symmetry by diagonalizing the transfer matrix. Algebraic BA works upon highest-weight representations.
We write down the $L_{an}^{(l,l)}$ matrix for this symmetries, with $l = 2s$

$$L_{an}^{(l,l)} = \begin{pmatrix} \sinh(\lambda + \frac{i\mu}{2} + i\mu S^z) & \sinh i\mu S^- \\ \sinh i\mu S^+ & \sinh(\lambda + \frac{i\mu}{2} - i\mu S^z) \end{pmatrix}. \tag{4.137}$$

It is possible to note that we have chosen the same λ for each spin chain site. It is possible (we shall see in the following sections) to have a spin chain with all the site with different λ. The hamiltonian of this "inhomogeneous" chain is, of course, more complicated than the usual XXZ one.
The first step is to define the *reference site-state* w,

$$|w\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.138}$$

annihilated by S^+

$$S_n^+|w\rangle_n = 0, \tag{4.139}$$

with $n = 1, \cdots, N$ and the *reference state* $|\Omega\rangle$

$$|\Omega\rangle = |w\rangle_1 \otimes \cdots |w\rangle_N. \tag{4.140}$$

Then, applying the monodromy matrix on (4.140) we obtain

$$T^{(l)}(\lambda)|\Omega\rangle = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} |\Omega\rangle, \tag{4.141}$$

where $\mathscr{A} = A_1 A_2 \cdots A_N$ and the same for the others three terms. When we take the trace, only the term $\mathscr{A} + \mathscr{D}$ is needed. We know from (4.139) that

$$A_n|w\rangle_n = \sinh(-\lambda + \frac{i\mu}{2} + i\mu s_n)|w\rangle_n, \tag{4.142}$$

where s_n is the spin of the n-site particle. We have assumed that every particle has the same spin. So $s_n \equiv s$.
We can be sure that

$$t_l^{(l)}(\lambda)|\Omega\rangle = \left[\left(\sinh(-\lambda + \frac{i\mu}{2} + i\mu s) \right)^N + \left(\sinh(-\lambda + \frac{i\mu}{2} - i\mu s \right)^N \right] |\Omega\rangle, \tag{4.143}$$

Following the proportionality of (4.132) and the definition (4.140), we can make the following *Bethe ansatz*: a general method to define a state is

$$|\Psi\rangle = \mathscr{B}(\lambda_1) \cdots \mathscr{B}(\lambda_M)|\Omega\rangle, \tag{4.144}$$

with $M \leq N$. Now we have to use the commutators between $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$, obtainable from the FRT (4.127), to study the action of $t_l^{(l)}(\lambda) = \mathscr{A}(\lambda) + \mathscr{D}(\lambda)$ on $|\Psi\rangle$. This gives something proportional to $|\Psi\rangle$ plus other terms. If the coefficients of these other unwanted terms go to zero , we can claim to have found an eigenstate of $t_l^{(l)}(\lambda)$.

These constraints on the unwanted coefficients produce the so-called *Bethe ansatz equations* (BAE). For instance, for the XXZ spin chain, we find that the BAE are

$$\left(\frac{\sinh(\lambda_i + i\mu s)}{\sinh(\lambda_i - i\mu s)} \right)^N = \prod_{i \neq j}^{M} \frac{\sinh(\lambda_i - \lambda_j + i\mu)}{\sinh(\lambda_i - \lambda_j - i\mu)}. \tag{4.145}$$

The solutions λ_i are called *Bethe roots* of the model. If we construct a state like in (4.144) using as λ_i the possible sets of Bethe Roots, we are sure to construct eigenstates of $t_l^{(l)}(\lambda)$ and therefore of H. The completeness of such eigenstate set can be proven, although we do not address here this problem (which is mathematically hard).
Hereafter, we denote $t_i \equiv t_i^{(i)}$.

4.6 Sine Gordon model.

The SG model is described by the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2\beta^2}\cos(\beta\phi), \tag{4.146}$$

where $\beta^2 \in [0, 8\pi]$ and m is the mass of the soliton (equal to the anti-soliton one). This Lagrangian is completely solvable and admits non-perturbative solutions, called *soliton* and *anti-soliton*. They are described by the fundamental representation of $SU(2)$. This solutions represent solitary waves in a viscous fluid which maintain the same speed and the same amplitude forever. The solitons are well-known objects in Fluid Dynamics: they were observed and described for the first time in 1834 by John Scott Russel and nowadays they are used in a very large number of physical and technical applications ranging from Fluid Dynamics to General Relativity to Non-linear optics.
The S-matrix between solitons s^+ and anti-solitons s^- can be factorized in two-body S-matrix satisfying YB equations. Its form has been found by A. Zamolodchikov and Al. B. Zamolodchikov [37] and can be written as

$$\mathbf{S}(\theta) = \frac{S_0(\theta)}{a(\theta)} R(\theta), \tag{4.147}$$

where

$$S_0(\theta) = e^{i\chi(\theta)} \qquad\qquad \chi(\theta) = \int_{-\infty}^{\infty} \frac{dk}{k} e^{ik\theta} \frac{\sinh[(p-1)k]}{2\sinh\frac{pk}{2}\cosh\frac{k}{2}}, \tag{4.148}$$

with $p = \frac{\beta^2}{8\pi - \beta^2}$ and β is the SG parameter. We remember that θ is the difference of rapidity between the two particles that scatter. The $S_{+-}^{+-}(\theta)$ is the transmission amplitude $S_T(\theta)$ and the $S_{+-}^{-+}(\theta)$ is the reflection amplitude $S_R(\theta)$.
The R-matrix is

$$R(\theta) = \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c & 0 \\ 0 & c & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix} \tag{4.149}$$

with

$$\begin{aligned} a(\theta) &= \sinh[(i\pi - \theta)/p], \\ b(\theta) &= \sinh(\theta/p), \\ c &= \sinh i\pi/p. \end{aligned} \tag{4.150}$$

The R-matrix in (4.148) is equivalent to (4.123) if we substitute $\lambda = \theta/p$ and $\mu = \pi/p$, that is we can obtain the SG S-matrix from the XXZ spin 1/2 chain R-matrix, correspondent to a $U_q(SU(2))$ symmetry. In fact the SG model has an infinite set of conserved charges because of the invariance with respect to the quantum group $U_q(SU(2))$ transformations.
The link between XXZ spin 1/2 model and SG model is the parameter μ, because

$$q = \exp\left(i\frac{\pi}{p}\right). \tag{4.151}$$

From (4.147)-(4.151) it is possible to turn out that SG model can be divided in three parts:

1. $\beta^2 \in]0, 4\pi[$ or $p \in]0, 1[$; the model is *attractive* and the spectrum is composed by the soliton, the anti-soliton and their bound states, named *breathers*;

2. $\beta^2 = 4\pi$ or $p = 1$; the model is completely free and describe the Dirac fermion, from the equivalence between (4.147) and the *Thirring model*;

3. $\beta^2 \in]4\pi, 8\pi]$ or $p > 1$; the model is completely *repulsive* and the spectrum is composed only by the soliton and the anti-soliton.

In $p = 0$ ($\beta^2 = 0$) the model is not defined.
In the repulsive regime (4.147) represents the whole S-matrix of the theory. Otherwise, in the attractive regime there appear poles in the physical strip and we must take into account bound states, called *breathers* (hereafter they will label by $|b_a\rangle = |s^+s^-\rangle$, $a = 1, \cdots, n = \lfloor \frac{1}{p} \rfloor$[18]). In the general case, the particle content of the theory is

1. s^+, s^- of mass M for any $p > 0$;

2. b_a of mass $m_a = 2M \sin \frac{pa\pi}{2}$ with $a = 1, \cdots, n = \lfloor \frac{1}{p} \rfloor$ for any $p \in]0, 1[$

In the attractive regime we have to add to the solitons S-matrix also the soliton-breather S-matrix [37]

$$S_a(\theta) = \frac{\sinh\theta + i\cos\frac{pa\pi}{2}}{\sinh\theta - i\cos\frac{pa\pi}{2}} \prod_{r=1}^{a-1} \frac{\sin^2\left(\frac{a-2r}{4}p\pi - \frac{\pi}{4} + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{a-2r}{4}p\pi - \frac{\pi}{4} - i\frac{\theta}{2}\right)} \tag{4.152}$$

[18]We denote with $\lfloor x \rfloor$ the integer part of x

and the breather-breather S-matrix

$$S_{ab}(\theta) = \frac{\sinh\theta + i\sin\left(\frac{a+b}{2}p\pi\right)}{\sinh\theta - i\sin\left(\frac{a+b}{2}p\pi\right)} \frac{\sinh\theta + i\sin\left(\frac{a-b}{2}p\pi\right)}{\sinh\theta - i\sin\left(\frac{a-b}{2}p\pi\right)} \times \prod_{r=1}^{\text{min(a,b)-1}} \frac{\sin^2\left(\frac{b-a-2r}{4}p\pi + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{b-a-2r}{4}p\pi - i\frac{\theta}{2}\right)} \frac{\cos^2\left(\frac{b+a-2r}{4}p\pi + i\frac{\theta}{2}\right)}{\cos^2\left(\frac{b+a-2r}{4}p\pi - i\frac{\theta}{2}\right)}. \tag{4.153}$$

4.6.1 Bethe-Yang equations.

For a quantum system with many particles and non-purely elastic scattering S-matrix we need the Bethe-Yang (BY) ansatz to find momentum eigenvalues [62]. If N is the total particle number, N_s is the number of solitons and N_a is the number of breathers of species a, of course $N = N_s + \sum_a N_a$. We put the N particles in a box of dimension[19] l with rapidities $(\theta_1, \cdots, \theta_{N+1})$, taking care to distinguish BY for the solitons from those for the breathers.
BY ansatz for the a soliton scattering with N_s solitons and $\sum_b N_b$ breathers is

$$e^{-Ml\sinh\theta_i} = \prod_{b=1}^{\lfloor\frac{1}{p}\rfloor}\prod_{j=1}^{N_b} S_b(\theta_{ij}) \cdot \mathrm{Tr}_i \prod_{k=1,k\neq i}^{N_s+1} \mathbf{S}_{ik}(\theta_{ik}) \tag{4.154}$$

We have defined $\theta_{ij} \equiv \theta_i - \theta_j$. The labels under $\mathbf{S}$ and R matrices describe which particles are scattered.
BY ansatz for the a breather of species a scattering with N_s solitons and $\sum_b N_b$ breathers is

$$e^{-im_a l\sinh\theta_i} = \prod_{b=1}^{\lfloor\frac{1}{p}\rfloor}\prod_{j=1,(j\neq i \text{ if } b=a)}^{N_b\ (+1 \text{ if } b=a)} S_{ab}(\theta_{ij}) \cdot \prod_{k=1}^{N_s} S_a(\theta_{ik}) \tag{4.155}$$

[19]It's important to remember that when we apply the TBA method, we are on a *finite size* manifold, with one of its two dimensions gone to infinity. When we work with a spin chain, however, we have *another* size, that is the length of the chain, normally called l. When these two sizes will appear together, we shall keep attention on specifying their nature.

Using(4.147), eq. (4.154) can be written as

$$
\begin{aligned}
e^{-ip_s(\theta_i)l} &= \prod_{b=1}^{\lfloor \frac{1}{p} \rfloor} \prod_{w=1}^{N_b} S_b(\theta_{iw}) \cdot \prod_{j=1, j\neq i}^{N_s+1} \frac{S_0(\theta_{ij})}{a(\theta_{ij})} \,\mathrm{Tr}_i \prod_{k=1, k\neq i}^{N_s+1} R_{ik}(\theta_{ik}) = \\
&= \prod_{b=1}^{\lfloor \frac{1}{p} \rfloor} \prod_{w=1}^{N_b} S_b(\theta_{iw}) \cdot \prod_{j=1, j\neq i}^{N_s+1} \frac{S_0(\theta_{ij})}{a(\theta_{ij})} \,\mathrm{Tr}_i \begin{pmatrix} \mathscr{A}(\theta_i|\vec{\theta}) & \mathscr{B}(\theta_i|\vec{\theta}) \\ \mathscr{C}(\theta_i|\vec{\theta}) & \mathscr{D}(\theta_i|\vec{\theta}) \end{pmatrix} = \\
&= \prod_{b=1}^{\lfloor \frac{1}{p} \rfloor} \prod_{w=1}^{N_b} S_b(\theta_{iw}) \cdot \prod_{j=1, j\neq i}^{N_s+1} \frac{S_0(\theta_{ij})}{a(\theta_{ij})} \left(\mathscr{A}(\theta_i|\vec{\theta}) + \mathscr{D}(\theta_i|\vec{\theta}) \right).
\end{aligned} \tag{4.156}
$$

Here $\vec{\theta} = (\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_{N_s})$. The second part of the third r.h.t is obtained from (4.126) and (4.120)-(4.121) with $R \equiv L$ because of the analogy between (4.132) and the first and the third expressions of (4.156).
We are evaluating the scattering of a test-particle with N other particles, each one being a particle of the theory. So far, we have already found the BAE for the $U_q(SU(2))$ invariant 1/2-representation (4.145); the only difference in this problem is that λ (now θ) is different for every sites. So, we are dealing with a completely inhomogeneous XXZ spin 1/2 chain. The transfer matrix is

$$
t_1(\theta_i|\vec{\theta}) = \mathrm{Tr}_i \prod_{k=1, k\neq i}^{N_s+1} R_{ij}(\theta_{ij}). \tag{4.157}
$$

where 1 labels the representation (in this case 1/2) of the auxiliary space. Following section 4.5.2, we define $|\Psi\rangle$ like in (4.144). Substituting $\lambda \equiv \theta/p$ and $\mu \equiv \pi/p$, we find

$$
\begin{aligned}
\mathscr{A}(\lambda_0|\vec{\lambda}, \vec{u})|\Psi\rangle &= \prod_{l=1}^{M} \frac{\sinh(\lambda_0 - u_l - i\mu)}{\sinh(\lambda_0 - u_l)} \prod_{k=1}^{N_s} \sinh(\lambda_k - \lambda_0 + i\mu)|\Psi\rangle + (\cdots), \\
\mathscr{D}(\lambda_0|\vec{\lambda}, \vec{u})|\Psi\rangle &= \prod_{l=1}^{M} \frac{\sinh(\lambda_0 - u_l + i\mu)}{\sinh(\lambda_0 - u_l)} \prod_{k=1}^{N_s} \sinh(\lambda_k - \lambda_0)|\Psi\rangle + (\cdots),
\end{aligned} \tag{4.158}
$$

where now we have called $\lambda_i \to \lambda_0$ and put this label out of the product for notation convenience. Here $\vec{u} = (u_1, \cdots, u_M)$ and $\vec{\lambda} = (\lambda_1, \cdots, \lambda_{N_s})$. Of course

$$
t_1(\lambda_0|\vec{\lambda}, \vec{u})|\Psi\rangle = \mathscr{A}(\lambda_0|\vec{\lambda}, \vec{u}) + \mathscr{D}(\lambda_0|\vec{\lambda}, \vec{u}))|\Psi\rangle. \tag{4.159}
$$

If we want to have an eigenvalue of $|\Psi\rangle$ with respect to t_1, we must impose that $(\cdots) = 0$.

We observe that, since the test particle must be a particle of the system, its rapidity must belong to the set of rapidities of the physical particles. For that reason, if λ_0 belongs to the set of rapidities of the physical particles, we find that $\mathscr{D}(\lambda_0|\vec{\lambda},\vec{u}))|\Psi\rangle = 0$ and we call $\lambda_0 \equiv \lambda_t$. (4.159) becomes

$$t_1(\lambda_t|\vec{\lambda},\vec{u}) = \prod_{l=1}^{M} \frac{\sinh(\lambda_t - u_l - i\mu)}{\sinh(\lambda_t - u_l)} \prod_{k=1\neq t}^{N_s} \sinh(\lambda_k - \lambda_t + i\mu), \tag{4.160}$$

where $t = 1, \cdots, N_s$. We can now write (4.156) in a more suitable form

$$e^{-ip_s(\lambda_t)l} = \prod_{b=1}^{\lfloor \frac{1}{p} \rfloor} \prod_{w=1}^{N_b} S_b(\lambda_{iw}) \cdot \prod_{j=1\neq t}^{N_s} S_0(\lambda_t - \lambda_j) \prod_{i=1}^{M} \frac{\sinh(\lambda_t - u_i - i\mu)}{\sinh(\lambda_t - u_i)}. \tag{4.161}$$

Looking at (4.160), we have to ensure the finiteness of the transfer matrix. For this reason we have to impose that

$$\begin{aligned} &\prod_{i=1\neq l}^{M} \sinh(u_l - u_i - i\mu) \prod_{k=1}^{N_s} \sinh(\lambda_k - u_l + i\mu) + \\ &+ \prod_{i=1\neq l}^{M} \sinh(u_l - u_i + i\mu) \prod_{k=1}^{N_s} \sinh(\lambda_k - u_l) = 0, \end{aligned} \tag{4.162}$$

which becomes the Bethe Ansatz equations for our problem

$$\prod_{k=1}^{N_s} \frac{\sinh(\lambda_k - u_l + i\mu)}{\sinh(\lambda_k - u_l)} = - \prod_{i=1\neq l}^{M} \frac{\sinh(u_l - u_i + i\mu)}{\sinh(u_l - u_i - i\mu)}. \tag{4.163}$$

We can arrange our equations by the substitution $u_k = \tilde{u}_k + i\mu/2$. From the following definition

$$\begin{aligned} Q_0(x|\vec{u}) &= \prod_{j=1\neq i}^{M} \sinh(x - u_j), \\ \psi_0(x|\vec{x}) &= \prod_{k=1}^{N_s} \sinh(x - x_k), \end{aligned} \tag{4.164}$$

we obtain

$$\begin{cases} e^{-ip_s(\lambda_t)l} = \prod_{b=1}^{\lfloor\frac{1}{p}\rfloor}\prod_{w=1}^{N_b} S_b(\lambda_{iw})\cdot\prod_{j=1\neq t}^{N_s} S_0(\lambda_t-\lambda_j)\frac{Q_0(\lambda_t-g|\vec{u})}{Q_0(\lambda_t+g|\vec{u})}; \\ e^{-ip_a(\lambda_T)l} = \prod_{b=1}^{\lfloor\frac{1}{p}\rfloor}\prod_{j=1,(j\neq T \text{ if } b=a)}^{N_b} S_{ab}(\lambda_{Tj})\cdot\prod_{k=1}^{N_s} S_a(\lambda_{Tk}); \\ \frac{\psi_0(u_l+g|\vec{u})}{\psi_0(u_l-g|\vec{u})} = -\frac{Q_0(u_l+2g|\vec{u})}{Q_0(u_l-2g|\vec{u})} \quad \text{(BAE)}, \end{cases} \tag{4.165}$$

where $g = i\mu/2$.

4.6.2 Density of roots and TBA.

We want to perform the thermodynamic limit $N \to \infty$ in order to find TBA equations for SG model. Looking at (4.165), we find that the l.h.s. of BAE becomes infinite. We have to compensate this divergence, adding singularities in the r.h.s. This implies the celebrated *string hypothesis.* Roots are organized into strings, defined as

$$u_{j,a}^{(n)} = u_j^{(n)} + \frac{i\mu}{2}(n+1-2a), \qquad a = 1, \cdots, n. \tag{4.166}$$

$u_j^{(n)}$ are the *centers* of string of length n. If the l.h.s of the BAE is less then unity, it becomes 0 in the $N \to \infty$ limit. If it's larger then unity, it becomes infinite. The r.h.s. must have the same behavior . So, at least one u_i must be so that

$$u_i - u_l = \begin{cases} -i\mu & \text{if l.h.s.} \quad < 1 \\ i\mu & \text{if l.h.s.} \quad > 1 \end{cases} \tag{4.167}$$

Now, the product over all string becomes the product between all roots (imaginary or real) into a single string times all single-n-type string between all kind of strings

$$\prod_{l=1}^{M} = \prod_{n\in\mathcal{U}_p}\prod_{l=1}^{M_n}\prod_{a=1}^{n}. \tag{4.168}$$

So there are M_n strings of type (n) in a given state and the number of all possible strings belongs to $\mathcal{U}_p$, where p indicates the irreducible representation of $U_q(SU(2))$ at $q = e^{i\pi/p}$. Because of not every single components of a string is a Bethe root, when we give the density of roots $\nu_n(u)$ we have to take in consideration also the density of the *holes* $\overline{\nu}_n(u)$, so that the total number of possible states is $\nu + \overline{\nu}$.

We can rewrite (4.165) so that

$$\begin{cases} Ml\sinh\lambda_t = -i\sum_{b=1}^{\lfloor\frac{1}{p}\rfloor}\sum_{w=1}^{N_b}\log S_b(\lambda_{iw}) + \sum_{j=1\neq t}^{N}\chi(\lambda_t-\lambda_j) - i\log\frac{Q_0(\lambda_t-g|\vec{u})}{Q_0(\lambda_t+g|\vec{u})}; \\ m_a l\sinh\lambda_T = -i\sum_{b=1}^{\lfloor\frac{1}{p}\rfloor}\sum_{j=1,(j\neq T\ \text{if}\ b=a)}^{N_b}\log S_{ab}(\lambda_{Tj}) - i\sum_{k=1}^{N_s}\log S_a(\lambda_{Tk}); \\ -\log\frac{\psi_0(u_l+g|\vec{u})}{\psi_0(u_l-g|\vec{u})} + \log\frac{Q_0(u_l+2g|\vec{u})}{Q_0(u_l-2g|\vec{u})} = 0 \qquad \text{(BAE)}, \end{cases} \tag{4.169}$$

If we apply (4.166)-(4.168) to BAE we obtain that (omitting the $\vec{u}$ dependence)

$$\prod_{a=1}^{n}\frac{\psi_0(u_{j,a}^{(n)}+g)}{\psi_0(u_{j,a}^{(n)}-g)} = \prod_{k=1}^{N}\frac{\sinh(u_j^{(n)}-\lambda_k+ng)}{\sinh(u_j^{(n)}-\lambda_k-ng)}; \tag{4.170}$$

$$\begin{aligned} &\prod_{a=1}^{n}\frac{Q_0(u_{j,a}^{(n)}+2g)}{Q_0(u_{j,a}^{(n)}-2g)} = \\ &= \prod_{m\in\mathcal{U}_pi}\prod_{l=1}^{M_m}\frac{\sinh(u_j^{(n)}-u_l^{(m)}+g(n+m))\sinh(u_j^{(n)}-u_l^{(m)}+g(n-m))}{\sinh(u_j^{(n)}-u_l^{(m)}-g(n+m))\sinh(u_j^{(n)}-u_l^{(m)}-g(n-m))}\times \\ &\quad\times\prod_{k=\frac{n-m+2}{2}}^{\frac{n+m-2}{2}}\left[\frac{\sinh(u_j^{(n)}-u_l^{(m)}+2gk)}{\sinh(u_j^{(n)}-u_l^{(m)}-2gk)}\right]^2. \end{aligned} \tag{4.171}$$

While the second and the third equations of (4.165) can be obtained trivially from (4.166), (4.170) and (4.171), the first one becomes

$$e^{-ip(\lambda_t)l} = \prod_{b=1}^{\lfloor\frac{1}{p}\rfloor}\prod_{w=1}^{N_b}S_b(\lambda_{iw})\cdot\prod_{k=1}^{N_s}S_0(\lambda_t-\lambda_k)\prod_{n\in\mathcal{U}_p}\prod_{l=1}^{M_n}\frac{\sinh(\lambda_t-u_l^{(n)}+ng)}{\sinh(\lambda_t-u_l^{(n)}-ng)} \tag{4.172}$$

In the thermodynamic limit ($N\to\infty$), we found that (4.172) can be written as

$$\nu_0(\lambda_t)+\bar{\nu}_0(\lambda_t) = Ml\cosh p\lambda_t + K_{0;\lambda}(\lambda_t-\lambda)*\nu_0(\lambda) + \sum_n K_{(n,l;\lambda)}(\lambda_t-u_l^{(n)})*\nu_{(n)}(u), \tag{4.173}$$

where $\nu_{(n)}(u) = \nu(u^{(n)})$ and convolutions are made with respect to: $d\lambda/2\pi$ in the first case; $du^{(n)}/2\pi$ in the second case. Furthermore, we define

$$\begin{cases} \nu_0(\lambda) + \overline{\nu}_0(\lambda) = \dfrac{n}{\Delta\lambda} & \text{(density of permitted rapidity);} \\ K_{0;y}(x) = \dfrac{d}{dy}\chi(x); & \\ K_{(n,l;y)}(x_l) = \dfrac{1}{i}\dfrac{d}{dy}\displaystyle\sum_{l=1}^{M_n} \log\left[\frac{\sinh(x_l + ng)}{\sinh(x_l - ng)}\right]; & \\ \nu_{(n)}(\lambda) \equiv \nu(\lambda^{(n)}) = \dfrac{n_{(n)}}{\Delta\lambda} & \text{(density of permitted center of string).} \end{cases} \tag{4.174}$$

Equivalently, we can find that BAE can be organized in the following manner: $\forall n$

$$\nu_{(n)}(u) + \overline{\nu}_{(n)}(u) = 2\sum_m K_{(m,l|n)}(u_l^{(m)} - u^{(n)}; \lambda, \lambda - u^{(n)}) * \nu_{(m)}(u), \tag{4.175}$$

where the convolution is made with respect to $du^{(n)}/2\pi$. We have defined $K_{(m,l|n)}(x_l; y, z)$ as

$$\begin{aligned} K_{(m,l|n)}(x_l; y, z) &\equiv K_{(-n-m,l;u^{(n)})}(x_l) + K_{(-|n-m|,l;u^{(n)})}(x_l) + \\ &+ 2\sum_{k=\frac{n-m+2}{2}}^{\frac{n+m-2}{2}} K_{(-2k,l;u^{(n)})}(x_l) + \frac{d}{du^{(n)}}\int dy K_{(-n,0;y)}(z)\nu_0(y). \end{aligned} \tag{4.176}$$

Summarizing, (4.169) can be written in the string approximation as

$$\begin{cases} \nu_0(\lambda_t) + \overline{\nu}_0(\lambda_t) = Ml\cosh p\lambda_t + K_{0;\lambda}(\lambda_t - \lambda) * \nu_0(\lambda) + \\ \qquad\qquad + \displaystyle\sum_n K_{(n,l;\lambda)}(\lambda_t - u_l^{(n)}) * \nu_{(n)}(u), \\ \nu_{(n)}(u) + \overline{\nu}_{(n)}(u) = 2\displaystyle\sum_m K_{(m,l|n)}(u_l^{(m)} - u^{(n)}; \lambda, \lambda - u^{(n)}) * \nu_{(m)}(u) \text{ (BAE).} \end{cases} \tag{4.177}$$

Minimizing the free energy, as explained in section 4.3 we are able to find the TBA in its universal form for the SG model, which is

$$\begin{cases} \delta_a^1 mR\cosh\theta = \epsilon_a(\theta) + \dfrac{1}{2\pi}\displaystyle\sum_{b=1}^{n} G_{ab}\left(\phi * L_b\right)(\theta), \\ \phi = \dfrac{1}{\cosh\theta}, \end{cases} \tag{4.178}$$

with the adjacency matrix of D_n Lie Algebra. This is the universal form which corresponds to $A_1 \diamond D_n$ with the mass on the first node.

SG

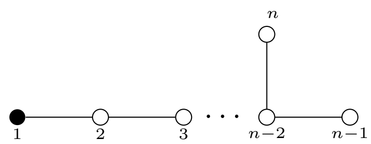


Figure 4.4: Dynkin diagram from the SG TBA (4.178).

4.6.3 Y-system for general scattering.

It has been noticed that the proof of classification of Y-systems for purely elastic scattering theories is absolutely independent of h. Other choices of the parameter h, where it no more plays the role of Coxeter number, can, in principle, lead to sensible TBA systems, like that in (4.107). The Y-system is the same as in (4.93), but now we define

$$Y_a(\theta) \equiv \exp(-\epsilon_a(\theta)). \tag{4.179}$$

for the magnonic particles. The conformal dimension Δ of the perturbing operator can be deduced from the periodicity of the Y-system and is independent on the choice of the particular nodes where masses or left-right movers are put. Of course, as the role of h is changed, the periodicity also gets some modification. (4.99) still holds, but now

$$P \equiv \frac{h+2}{2}. \tag{4.180}$$

It is possible to find, for each Lie Algebra, the dimension of the corresponding perturbing operator:

$$\begin{cases} A_n: \quad \Delta = 1 - \dfrac{2}{n+2}; \\ D_n: \quad \Delta = 1 - \dfrac{1}{n}; \\ E_6: \quad \Delta = \dfrac{6}{7}; \\ E_7: \quad \Delta = \dfrac{9}{10}; \\ E_8: \quad \Delta = \dfrac{15}{16}. \end{cases} \tag{4.181}$$

4.7 Integrable perturbations of the Z_n parafermion models and the $O(3)$ NLSM.

Here we focus on the CFT which gives origin to this model from a particular perturbation. This CFT is the Z_n parafermion models. we shall see that, if $n \to \infty$, we shall obtain an infinite TBA system describing the same finite-size effects of the $O(3)$ NLSM in the UV limit.
In 2d QFT, fields can have different spin. In addition to integer and semi-integer spin, there exist semi-local fields with rational spin, and we call these fields *parafermions*.

The series of Z_n, $(n \geq 2)$ CFT parafermion models was discovered in [61]. From the CFT point of view Z_n corresponds to the Virasoro central charge $c = \frac{2(n-1)}{n+2}$. The characteristic parafermion symmetry, which includes the conformal one as a sub-symmetry, is generated by the set of parafermion currents $\Psi_k(z)$, $k = 1, 2, \cdots, n-1$, with "right" dimensions $\Delta_k = k(n-k)/n$ (of course, there is also the same "left" symmetry, generated by the left currents $\overline{\Psi_k}(\overline{z})$). The algebra of the parafermion currents, OPEs and structure constants are described in detail in [61].
Following the same paper, we denote here the corresponding non-critical model as H_n. The perturbed action under consideration reads

$$A_{H_n} = A_{Z_n} - \int d^2x \left(\lambda \Psi \overline{\Psi} + \lambda \Psi^+ \overline{\Psi}^+\right), \tag{4.182}$$

where $\lambda > 0$, $\lambda \sim [\text{mass}]^{2/n}$.This perturbation is integrable and the massive theory H_n is factorizable.

If $n = 2$, Z_2 is the critical Ising CFT with $c = 1/2$ and, being Ψ a Majorana fermion, H_2 is the temperature perturbation of the critical Ising, that is the theory of free massive Majorana fermions.
Z_3 is the critical three-states Potts model, with $c = 4/5$. H_3 is the factorized RSOS scattering theory.
Z_4 is equivalent to the sine-Gordon model at $\beta^2 = 6\pi$ and H_4 fields correspond to solitons.
In the limit for n tends to ∞, we find that fields form an $O(3)$ triplet and the corresponding FST turns to coincide with that of the $O(3)$ NLSM.

Here we deal with a case of non-diagonal scattering. Differently from the derivation in paragraph 4.3, one has to apply the higher-level Bethe ansatz technique ([62]), introducing pseudo-particles (what we have already called magnons) to take into account the color structure of the Bethe wave function. The TBA system has the same form (4.58), but now some of the pseudo-energies are related to the magnons or their bound states, not

to real particles. Since the magnons carry no energy and momentum, the corresponding energy terms in the TBA equations $m_a R \cosh\theta = \nu_a$ are zero. In particular, the magnons do not contribute to the Casimir energy (4.63).
Unfortunately, nowadays it doesn't exist a general method to derive TBA equations from non-diagonal FST. Each case needs a particular analysis. The best approach to this problem is to pass to the universal Y-system and try to search a Dynkin diagram which could be used to reproduce the finite-size effects expected. If the universal function $\phi_{ab}(\theta) = G_{ab}\phi$ is given ((4.107), we could reproduce the corresponding Dynkin diagram.

The adjacency matrix for Z_2 correspond to $A_1 \oplus A_1 = D_2$ Lie algebra; that of Z_3 to $A_3 = D_3$; that of Z_4 to D_4. We anticipate here that the SG model for $\beta^2 = 8\pi(n-1)/n,\ n = 3, 4, ...$ corresponds to D_n but with the mass on the tail and not on the fork. We could expect that Z_n correspond to the Dynkin diagram D_n, with the mass on the fork, to distinguish it from the SG model. The $O(3)$ model could be reproduced by a Dynkin diagram D_N with $N \to \infty$.
It is possible to check this assumption in many ways. We prefer to refer the interested reader to the original paper [61]. We only specify an interesting result that will become useful in the next chapter.
In the UV limit, the finite-size scaling function C tends to $C_{UV} = 2$. Near this limit we can assume that this function is near 2, with a little perturbative discrepancy

$$c(r) = 2 - e_{PF}(r, n) + \cdots \tag{4.183}$$

where, to compare the Z_n c_{UV} with that of the $O(3)$ NLSM we must perform the $n \to \infty$ limit. Now, it is possible to show that

$$e_{PF}(r, n) = \frac{6}{n+2} + \sum_{k=1}^{\infty} b_k(n) e^{-4k\tau/n}, \tag{4.184}$$

where b_k are some coefficients and $\tau \equiv -\log mr$.
In the UV limit, the main contribution to e_{PF} comes from $0 < k < n/\tau$, where

$$b_k(n) = \frac{12}{n}\left(1 + O(k\log(k/n)/n)\right), \tag{4.185}$$

from which

$$e_{PF}(r, n) = \frac{6(1 + e^{-4\tau/n})}{n(1 - e^{-4\tau/n})} + O(\log n/n). \tag{4.186}$$

In the $n \to \infty$ limit, we have

$$\lim_{n\to\infty} e_{PF}(r, n) = \frac{3}{\tau} + O(\log\tau/\tau^2). \tag{4.187}$$

It turns out that the universal TBA or Y-system for this model is that described in section 4.4.4 with the Simple Lie algebra D_∞.

$O(3)$ NLSM

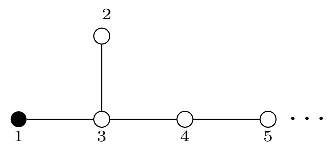


Figure 4.5: Dynkin diagram of $O(3)$. We can observe the great difference between this model and SG (Fig 4.4). $O(3)$ has the massive node on the fork, while SG model has the black node on the tail.

Chapter 5

Integrable deformations of the $O(3)$ NLSM: sausage model.

This chapter is entirely devoted to a one parameter family of models that can be considered the integrable deformation of the $O(3)$ NLSM.
We call this deformation *the sausage model* because the undeformed model, the $O(3)$ NLSM, has a spherical metric and the deformation changes it, stretching the sphere into an axially symmetric shape remembering the form of a sausage, with two almost semi-spherical edges and a "cylindric" body. This metric is characterized by a $U(1)$ symmetry.
The deformation parameter, which we call λ, is such that, for $\lambda \to 0$, the metric becomes a perfect sphere and the $O(3)$ NLSM is recovered.

The deformation is built in such a way to preserve integrability of the model. The sausage is, therefore, an integrable system, with a factorizable block-diagonal S-matrix, TBA equations and associated Dynkin diagram. Hidden symmetries exist, like in $O(3)$ (see 2.3.3) or SG (see [71]). They are non-linear symmetries related to *quantum groups*, that is, deformation by a parameter of *classical groups*, like $SU(2)$ or $SO(3)$[1].

First, we give a brief definition of the Renormalization Group (RG) and its flow through the coupling space. We shall see how the renormalization scheme works both in QFT and in statistical mechanics. We also take a look to the properties of 2d metric and curvature in relation with the RG.

Second, following the original papers of Fateev, Onofri ad Zamolodchikov ([64]-[66]), we shall study the Sausage model, initially defining its S-matrix from the symmetries of the problem, then, trough the RG flow, we shall go in the conformal regime to study the UV properties of the corresponding QFT.

[1]The name "quantum group" is somewhat misleading. Actually, we are considering algebras, not groups. We shall denote a one parameter (q) deformation of the (universal enveloping of the) algebra $\mathfrak{g}$ by $U_q(\mathfrak{g})$.

5.1 Preliminaries.

To understand better how to construct the sausage structure mentioned above, we shall give some remarks on RG, 2d geometry and stress some peculiarities of 2d RG.

5.1.1 Renormalization Group and RG flow.

Both in canonical and path integral approach, QFT has the great problem of infinities: at first sight, physical quantities often turn out to be represented by divergent integrals. To solve this problem, from the early years of QFT physicists have introduced cut-off techniques in order to obtain finite integrals. Time passing, these techniques became more and more sophisticated and physically motivated.
Nowadays, we know that the most useful tool to deal with infinities is the *renormalization*, that is, we add a finite number of divergent terms to the Lagrangians, called *counter terms*, which compensate the infinite terms arising in perturbation expansion.
These terms are integrals in the momentum space which diverge for $p \to \pm\infty$. With the *cut-off* technique it is possible to make the integration convergent. In fact, substituting the integral endpoints $\pm\infty$ by a finite term $\pm\Lambda$, physical quantities don't become infinite, but they depend on Λ. This dependence must disappear at the end of calculation for physical quantities. For dimensional consistence it must be dimensionless and this fact leads to the introduction of an arbitrary scale of mass, μ. To be physically consistent, physical quantities must be independent from Λ/μ. To obtain this, coupling constants α_i can actually vary depending on the scale μ and must respect the *beta function* $\beta^i(\alpha)$ equation

$$\frac{d\alpha^i}{d\log\mu} = \beta^i(\alpha), \tag{5.1}$$

where $\alpha = (\alpha^1, \cdots, \alpha^n)$ is the set of the coupling constants of the theory. If (5.1) holds, for $\Lambda \to \infty$ we recover the theory without divergences.
If we a have a Lagrangian with more terms, or a massive field, or more fields interacting each other, we must introduce more renormalization terms, that we denote with Z and a label i which specifies to what Lagrangian terms is referred (for instance, Z_m is attached to the massive term).
Equations that tell us how the Lagrangian parameters vary with μ are collectively called the equations of the Renormalization Group (RG). It is not a true group of transformation, but a semi-group[2]. This property is easy to

[2]This fact is related to the irreversibility of the RG transformation.

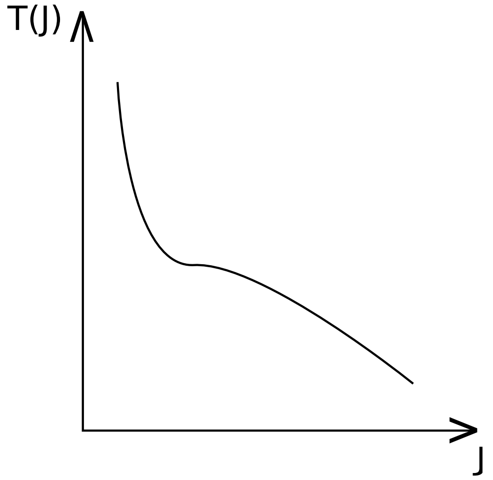


Figure 5.1: An example of RG flow.

understand from the statistical mechanics point of view.

In this case, we can touch with hand what is the deep meaning of the RG action. Taking a lattice theory defined by an Hamiltonian that depends on the interaction between near sites, for instance

$$H = -J \sum_{<ij>} < \sigma_i \sigma_j >, \tag{5.2}$$

where $< ij >$ defines the correlation length of the theory, σ are the angular momentum operators and J is a coupling parameter. Each σ_i acts only on the i-th site.
If we want to take a more general look on the system, we have to rescale the system, defining a bigger correlation length and calculating correlation functions in the new scale. This operation is quite similar to consider the thermodynamics of a system, because we consider a block-Hamiltonian, where each block is the result of many single interactions. This new block-Hamiltonian can be called H' and depends on new coupling parameter J', that we call *effective* parameter. The RG is the set of equations that drive the system from $H(J)$ to $H'(J')$ and it's not reversible.

We call the RG *trajectory or flow* the function $T(J) = J'$ in the space of the coupling parameter. (Fig.(5.1))

If we have a criticality, for instance if we are in the UV regime of a QFT, so in its "CFT counterpart", the correlation length becomes infinite. For this reason, in this point we have $T(J_c) = J_c$, that is, we have a *fixed point* in the space of the configuration parameter J. We can exit from this point to study the RG flow of a CFT into a QFT only if we perturb the system, as explained in section 4.4.

We note that a rescaling of the system is made, in 2d, by the transformation $x' = \rho x$, where ρ is the scaling parameter. If $\rho > 0$, then we have a new system, bigger than the old one. Now, it is possible to find that $\mu = 1/\rho$, where μ is defined in (5.1). We could say that the possibility to rescale the system is needed in order to cancel infinities at the quantum level.
In a fixed point, the beta function is zero.

5.1.2 2d world.

In order to understand better what will follow in the next chapters, we briefly recall some properties of the Riemann tensor and of the RG flow in 2d systems.

Riemann tensor.

In two dimensions, the Riemann tensor has only one independent component, which can be taken as R_{1212}. Defining g as the determinant of the metric tensor $g_{\mu\nu}$, we can obtain:

$$\begin{aligned} R_{\mu\nu} &= g_{\mu\nu}\frac{R_{1212}}{g} \qquad && \text{Ricci tensor;} \\ R &= 2\frac{R_{1212}}{g} && \text{curvature scalar;} \end{aligned} \tag{5.3}$$

This will be useful to derive the RG evolution equation for the Sausage model.

Zamolodchikov c-theorem.

For each unitary field theory in two dimensions there exists a function $c(\alpha)$ such that [63]:

1. it is decreasing along the RG flow
$$\frac{d\ c}{d\log\mu} \equiv \sum_i \beta^i(\alpha)\frac{\partial}{\partial\alpha^i}c(\alpha) \leq 0; \tag{5.4}$$

2. $c(\alpha)$ is stationary in $\alpha = \alpha_*$ if and only if α_* is a fixed point of the RG flow, i.e. $\beta(\alpha_*) = 0$;

3. in these fixed points, $c(\alpha)$ completely defines the two-point correlation function of the energy-momentum tensor $T_{\mu\nu}$.

From the third point it is clear that this function is the central charge of the theory.
Thinking to a theory defined by an action defined upon some RG flow connecting an UV fixed point to an IR one, the c-theorem can be summarized by the inequality

$$c_{UV} \geq c_{IR}. \tag{5.5}$$

It is found also that

$$\beta^i = -\sum_j \frac{1}{12} g^{ij}(\alpha) \frac{\partial c(\alpha)}{\partial \alpha^j}, \tag{5.6}$$

where $g_{ij}(\alpha)$ specifies the metric in the coupling space.

5.2 The Sausage model.

We recall that the action of a generic NLSM where ϕ belongs to a Riemannian manifold $\mathscr{M}$ is

$$A[g] = \frac{1}{2} \int d^2x g_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j, \tag{5.7}$$

where (x^0, x^1) are the coordinates of a flat space-time and $g_{ij}(\phi) = g_{ji}(\phi)$ is the metric tensor of the d-dimensional manifold[3].
We have already discuss the renormalizability of the theory for g_{ij} not far from flat η_{ij} (hereafter we write the flat metric in euclidean coordinates, i.e. δ_{ij}, by invoking a Wick rotation) . The remarkable point here is that the 1-loop RG evolution can be calculated [69] to be

$$\frac{d}{d \log \mu} g_{ij} = -\frac{1}{2\pi} R_{ij} + O(R^2). \tag{5.8}$$

For some value of μ we can have a strong value of g_{ij}. In this case the theory can't be studied perturbatively and our analysis would fail. In particular, for $\log \mu \to \pm\infty$ (or in the UV and in the IR limit respectively), g_{ij} could become too far from δ_{ij}.

Fortunately, we shall be in a special case, in which $\mathscr{M}$ is an *Einstein manifold*, i.e. a Riemann manifold with $R_{ij} = K g_{ij}$. For instance, the Hypersphere S^n, the euclidean space, the hyperbolic space (de Sitter or Anti-de Sitter), CP^n, Calabi-Yau spaces are all Einstein manifolds. For these special (but not exclusive!) manifolds, for $\mu \to 0$, g_{ij} doesn't become too big to drop the perturbative approach. We can say that in the UV limit the

[3]We have chosen the greek letter for the auxiliary space (world-sheet in the string language) and the latin letters for the target space (usual space-time in the string language).

theory continues to be perturbative. *Au contraire*, in the IR limit the metric grows until it breaks the perturbative approach: we must find something else to study the model in this case.

Our desire is to find a quantum integrable model which in some limit becomes the $O(3)$ NLSM and which continues to be integrable. We shall start from a plausible FST, then we compare its UV behavior with the results in the same regime of the perturbative expansion of a field theory on an Einstein manifold. We shall find an indirect evidence of *a*) integrability, *b*) scattering theory of the field theory. It is known that some trajectory of (5.8) is integrable. This means that, starting from $\mu \sim 0$, the metric changes with μ but the model remains integrable. We shall find one of these flows.

In the IR limit, where the perturbative approach drops out, we can use the powerful methods of CFT. In fact, the IR limit is a fixed point of the RG flow and this permits us to study this point, out of perturbative analysis, using "conformal methods". We could compare IR limit results in both FST and field theory, finding non trivial coincidences.

We start analyzing the RG flow of a NLSM where fields take value on S^2, so we rewrite the action of the $O(3)$ NLSM

$$A[g] = \frac{1}{2} \int d^2x g_{ij} \partial_\mu \phi^i \partial_\mu \phi^j, \tag{5.9}$$

where now g_{ij} is the metric on the 2d sphere.
One can always choose, at least locally, conformal coordinates $x = (X, Y)$[4] on $\mathscr{M}$, for which

$$g_{ij}(x) = e^{\Phi(x)} \delta_{ij}, \tag{5.10}$$

from which, with the help of (5.8) and [69]-[70] we find the *RG evolution equation*

$$-\frac{d\Phi}{d \log \mu} = \frac{1}{4\pi} R + \cdots \tag{5.11}$$

We stress here that the topology of $\mathscr{M} = S^2$ will be conserved through the entire RG flow.
It is easy to find that

$$\frac{d}{d \log \mu} e^{\Phi} = \frac{1}{4\pi} \triangle \Phi. \tag{5.12}$$

In the UV direction solutions of (5.12) are usually unstable. Nevertheless, we shall find a one-parameter family of axially symmetric solutions with stable UV behavior. These solutions exhibit a monotonically growing manifold. If we define the volume of $\mathscr{M}$, $V_{\mathscr{M}} = \int \sqrt{g} d^2x$, we find from (5.12) that $V(\mu) = -2(\log \mu - \log \mu_0)$, then

[4] $0 \leq X \leq 2\pi$, $-\infty \leq Y \leq \infty$

1. for $\mu \to 0$, $V \to \infty$ and the metric becomes the euclidean space, without curvature;

2. for $\mu \to \infty$, the curvature becomes infinite and the perturbative calculation "shrinks", because $V \to 0$ in $\mu = \mu_0$.

From section 3.3.2 we have the $O(3)$ NLSM action in a more suitable form. We have just seen that the $O(3)$ NLSM is *asymptotically free*, i.e. the coupling decreases while the energy increases. An example of a famous asymptotically free theory is QCD. The behavior of g is to decrease logarithmically with μ,

$$g \sim -\frac{2\pi}{\log \mu}. \tag{5.13}$$

We know from section 3.3 the S-matrix of the $O(3)$ NLSM. The spectrum is composed by a $O(3)$-triplet of massive particles and the Y-system is referred to a D_n Dynkin diagram, with the mass on the fork. The central charge in the UV limit is $c_{UV} = 2$.

Here comes the most important point of our analysis: if we ask to a FST the $U(1)$ symmetry, we find a one parameter (λ) family that, at $\lambda = 0$, becomes the FST of $O(3)$ NLSM, enlarged by the $SU(2)$ symmetry (we remember that $U(1) \subset SU(2)$). We call these FST, SST_λ^{+}[5], i.e. Sausage Scattering Theory. We shall see this FST in section 5.2.1. In section 5.2.2 we analyze the axially symmetric solutions of the RG flow (5.12) and we find a one parameter family (ν) that preserves the integrability. It will turn out that, at $\nu = 0$, the equivalent QFT is the $O(3)$ NLSM. We suppose that each of these QFT's has a correspondent FST in the SST_λ^{+} family. We call these QFT's SSM_ν^{0}[6], i.e. Sausage Sigma Model, hereafter just "Sausage model". In section 5.2.3 we analyze the UV behavior of SSM_ν^0 family, finding a non trivial equivalence with SST_λ^{+}. See Fig.(5.2).

Finally, we explicit the more exciting point of these studies: that we have found a deformation of a quantum integrable system that have preserved the integrability.

5.2.1 Sausage scattering theory.

In addition to the usual constraints on the two-body S-matrix, we impose the $U(1)$ symmetry to the Yang-Baxter equation. We find a spectrum composed by 3 particles A_s, with $s = 0, \pm$, with the same mass m. The invariance

[5] The "+" is linked to the possibility of adding a topological term; we don't consider this case here and,in the following, we drop this label.

[6] The "0" is again linked to the possible topological term and we ignore this label too.

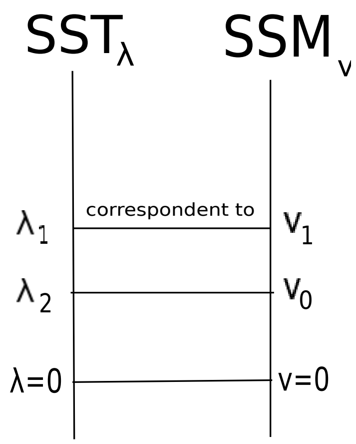


Figure 5.2: We see the one-to-one correspondence between FST SST_λ and SSM_ν. Every IQFT of the SSM_ν type is integrable with $c_{UV} = 2$. They converge at $\lambda = \nu = 0$ to the $O(3)$ NLSM.

with respect to $U(1)$ transformation can be written as the invariance of the S-matrix under the transformations:

$$A_0 \to A_0 \qquad A_\pm \to e^{\pm iQ\xi} A_\pm, \tag{5.14}$$

where $\xi \in [0, 2\pi[$ is the $U(1)$ angle and the total charge $Q_T = \sum_i Q_i = 0$ is conserved.
Other symmetries are: C (charge conjugation, $\overline{s} = -s$), P (space parity), T (time reversal). One finds

$$\begin{aligned}
S_{++}^{++}(\theta) &= S_{+-}^{+-}(i\pi - \theta) = \frac{\sinh\lambda(\theta - i\pi)}{\sinh\lambda(\theta + i\pi)}, \\
S_{+0}^{0+}(\theta) &= S_{+-}^{00}(i\pi - \theta) = -i\frac{\sin 2\pi\lambda}{\sinh\lambda(\theta - 2i\pi)} S_{++}^{++}(\theta), \\
S_{-+}^{+-}(\theta) &= -\frac{\sin\pi\lambda \sin 2\pi\lambda}{\sinh\lambda(\theta - 2i\pi)\sinh\lambda(\theta + i\pi)}, \\
S_{+0}^{+0}(\theta) &= \frac{\sinh\lambda\theta}{\sinh\lambda(\theta - 2i\pi)} S_{++}^{++}(\theta), \\
S_{00}^{00}(\theta) &= S_{+0}^{+0}(\theta) + S_{-+}^{+-}(\theta).
\end{aligned} \tag{5.15}$$

The appearance of λ is evident. These elements have a $i\pi/\lambda$ periodicity. If we take the limit $\lambda \to 0$, in the new base of states $A_1 = (A_+ + A_-)/\sqrt{2}$, $A_2 = -i(A_+ + A_-)/\sqrt{2}$, $A_3 = A_0$, we find the explicit S-matrix element for the $O(3)$ NLSM!

It is possible to divide the λ-space in three regions:

1. $\lambda \in [0, 1/2[$. We call this region *repulsive* because the only stable particles are $A_1,\ A_2,\ A_3$. This means that we haven't find any pole in the physical strip $0 \leq \mathrm{Im}(\theta) < \pi$.
2. $\lambda = 1/2$. In this point the theory becomes a free field theory of two identical fermions $A_\pm$ and a boson A_0 with the same mass m.
3. $\lambda > 1/2$. We call this region *attractive* because in the physical strip the S-matrix elements show a lot of poles, that means bound states.

We shall analyze here the simpler case, the repulsive one.

5.2.2 Sausage trajectories or the Sausage RG flow

If we impose to the solution to be axially symmetric, we shall find that (5.12) becomes

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = e^{-\Phi} \dfrac{1}{4\pi} \dfrac{\partial^2 \Phi}{\partial Y^2}, \\ \Phi(x) = \Phi(Y), \end{cases} \tag{5.16}$$

where $t \equiv \log \mu$. For $Y \to \pm\infty$, $\Phi(Y) \sim -2|Y|$. This imply the smoothness of the metric at the poles.
In this coordinates, we have, from (5.9), the *axially symmetric action*

$$A = \frac{1}{2} \int d^2x \left\{ e^{\Phi(Y)} \left[(\partial_\mu Y)^2 + (\partial_\mu X)^2 \right] \right\}. \tag{5.17}$$

We stress here that the $O(3)$ NLSM action is an action in the flow of (5.17). Here we follow all the solutions of the RG evolution equation.

To solve this equation, we need the following ansatz:

$$\Phi(Y) = -\log \frac{a(t) + b(t)\cosh(2Y)}{2}, \tag{5.18}$$

where $\Phi \in \mathbb{R}$ and non-singular if $b \geq 0$ and $a \geq -b$.
From (5.16) we find that

$$\frac{da}{dt} = \frac{1}{2\pi} b^2; \qquad \frac{db}{dt} = \frac{1}{2\pi} ab; \tag{5.19}$$

calling $\nu^2 = a^2 - b^2$, we find

$$\begin{aligned} a(t) &= -\nu \coth \frac{\nu(t - t_0)}{2\pi}, \\ b(t) &= -\nu \sinh^{-1} \frac{\nu(t - t_0)}{2\pi}. \end{aligned} \tag{5.20}$$

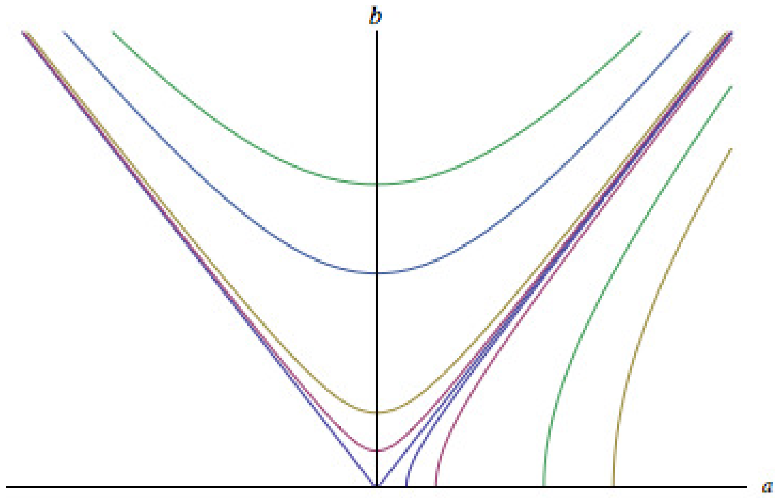


Figure 5.3: Hyperbolic trajectories from (5.20) in the $a - b$ plane.

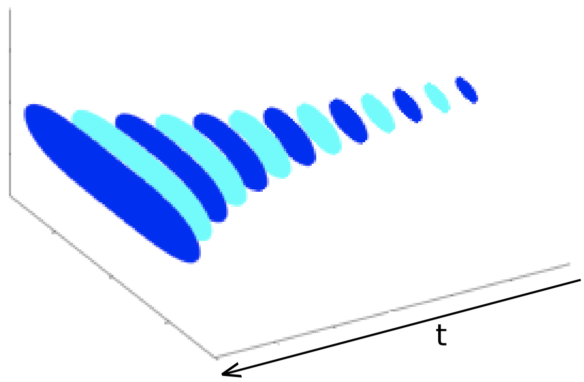


Figure 5.4: This is a numerical evolution of the metric (5.10) with (5.18) with respect to the parameter t. Starting from the $O(3)$ NLSM metric, the S_2 sphere, it becomes a sausage-like metric in the UV limit. This image is taken from [72].

All these trajectories are hyperbolic, as drawn in Fig.(5.3). We remember that $t_0 > t\ \forall t$. We are interested in the UV stable trajectories, which begin at the line of the UV fixed point $b = 0,\ a \geq 0$ and fill the sector $b \geq 0,\ a \geq b$.

We can see how the metric reacts to the evolution of these two parameters in Fig.(5.4). We plot for a little value of ν the evolution in the UV limit. The manifold, for $\nu(t - t_0) \ll 1$, appears like a sphere but as soon as the term $(t - t_0)$ starts to grow, $\mathscr{M}$ transforms into an elongated shape (that the authors of [64] call sausage) of length

$$L \sim \frac{\sqrt{2\nu}}{2\pi}(t_0 - t), \tag{5.21}$$

and width (far from the edges)

$$l = 2\pi\sqrt{2/\nu}. \tag{5.22}$$

We identify the point $a = \nu,\ b = 0$ with the CFT $\mathbb{R} \times S^1(l)$, where $S^1(l)$ is the circle of circumference l, given by 2 free bosons, one uncompactified ($\mathbb{R}$), the other compactified on $S^1(l)$. If also $\nu = 0$, we return in the $O(3)$ NLSM configuration!

It's important to note that $\nu \ll 0$, because only if this condition is respected we can have an anisotropic Sigma Model. In fact, if $\nu \to \infty$, immediately $a < -b$ and we don't have stable UV solutions. If $a \sim b$, the metric in this limit results

$$g_{ij} = \frac{\pi}{2}\frac{1}{\cosh^2 Y}\delta_{ij}. \tag{5.23}$$

Now we can write for each value of ν the respective QFT 1-loop renormalized action

$$A_{SSM_\nu} = \int d^2x \frac{(\partial_\mu Y)^2 + (\partial_\mu X)^2}{a(t) + b(t)\cosh(2Y)}. \tag{5.24}$$

This action becomes exact in the *scaling limit*, i.e.

$$\nu \to 0, \quad t \to -\infty, \quad \nu t = \text{const}. \tag{5.25}$$

It is worth mentioning that, in terms of unit field $\phi^2 = 1$, (5.24) becomes

$$\begin{cases} A_{SSM_\nu} = \dfrac{1}{2g(t)}\displaystyle\sum_{a=1}^{3}\int d^2x \frac{(\partial_\mu \phi_a)^2}{1 - \nu^2\phi_3^2/2g^2(t)}, \\ g(t) = \dfrac{\nu}{2}\coth\dfrac{\nu(t - t_0)}{4\pi}. \end{cases} \tag{5.26}$$

5.2.3 The hot Sausage (or the SSM UV limit).

Differently from the TBA analysis of chapter 4, we consider here the space (proportional to the temperature T) as the finite dimension and the time as the infinite dimension. If we continue to reduce the space we take in consideration larger and larger energies, going to the UV limit of the theory. Consider an infinite cylinder of circumference $r \sim 1/T$, where the time is put on infinite side and the space on the finite one. If our QFT is unitary and tends to some CFT in the UV limit, we have (see section 4.1.2)

$$E(r) = -\frac{\pi \tilde{c}(r)}{6r}, \tag{5.27}$$

where $\tilde{c}(0) = c_{UV} = 2$. We call $c_\nu(r)$ the corresponding finite-size scaling function of the SSM_ν.

We can start from this point to build a suitable Hamiltonian operator. It is possible to assume that the conformal coordinates, due to the strong curvature of the space dimension, have a part independent from the space coordinate x_2, which we call *zero mode part*

$$\begin{aligned} X(x_1, x_2) &= x(x_1) + \xi(x_1, x_2); \\ Y(x_1, x_2) &= y(x_1) + \eta(x_1, x_2). \end{aligned} \tag{5.28}$$

We remember that x_1 represents the imaginary true time. Of course, $\int \xi(\eta) dx_2 = 0$. Now, we know that:

1. $g^{ij} = e^{-\Phi(y(x_1))}\delta^{ij}$;
2. $\mathscr{L} = e^{\Phi(Y)}\left((\partial_\mu X)^2 + (\partial_\mu Y)^2\right)$;
3. $\mu \sim r$;
4. $e^{-\Phi}/2r$ is equal to ∞ if $t \to -\infty$ (UV) and to 0 if $t \to \infty$ (IR);
5. for generalized Hamiltonian it's better to use the Weyl (or symmetric) order (Indicated by a $S[\cdot]$ in the following expression).

So, we can ask to the Hamiltonian of SSM_ν to be

$$\begin{aligned} \hat{H} &= -\frac{1}{2r} S\left[e^{-\Phi(y)}\left((\partial_\mu x)^2 + (\partial_\mu y)^2\right)\right] = \\ &= \frac{1}{r} e^{\Phi(y)/2} \hat{h} e^{-\Phi(y)/2}, \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} \triangle &= e^{-\Phi(y)}\left(\partial_y^2 + \partial_x^2\right), \\ R &= -e^{-\Phi(y)}\partial_y^2\Phi(y) \end{aligned} \tag{5.30}$$

and R is the curvature scalar.
We can assume that

$$c_\nu(r) = 2 - e_\nu(r) + \cdots, \tag{5.31}$$

where

$$\hat{h}\Psi_0 = \frac{\pi e_\nu}{6}\Psi_0, \tag{5.32}$$

with Ψ_0 the ground-state of the model.

Now the problem becomes more and more complicated. We do not give the detailed history of the derivation of the following results, but only a brief review of the procedure used.
(5.32) can be transformed in

$$-\frac{1}{2}\Psi_0''(y) - \frac{1}{8}\Phi''(y)\Psi_0(y) = e^{\Psi(y)}\frac{\pi e_\nu}{6}\Psi_0(y). \tag{5.33}$$

In 1-loop approximation, the differential operator $\hat{h}/\nu$ depends only on the "scaling " combination $\eta = \nu(t_0 - t)/4\pi$. Therefore

$$e_\nu(r) = \frac{\nu k(\eta)}{4\pi}, \tag{5.34}$$

with $k(\eta)$ called *scaling function*[7]. The scaling function is the minimum eigenvalue of the Sturm-Liouville problem

$$-\left(-\cosh 2\eta + \cosh 2y\right)\partial_y^2\Psi_0 + \frac{1 + \cosh 2\eta \cosh 2y}{\cosh 2\eta + \cosh 2y}\Psi_0 = \frac{k(\eta)}{6}\sinh 2\eta\Psi_0. \tag{5.35}$$

This is another eigenvalue problem of the form

$$L\Psi_0 = k(\eta)\Psi_0. \tag{5.36}$$

It's useful to pass from this form to the *Lamè form* by using elliptical functions, and then calculate the IR and UV limits, but we prefer to omit this technical argument and to go directly to the results obtained.

[7] The significance of the word scaling is far from the finite-size scaling function C.

We can resolve analytically (5.36) only in the IR and UV limits that are equivalent, respectively, to $\eta \to 0$ and $\eta \to \infty$ limits. We obtain

$$\begin{aligned} &\text{UV)} \qquad k = \frac{3\pi^2}{2(\eta + 2\log 2)^2} + O(\eta^{-4}); \\ &\text{IR)} \qquad k = \frac{3}{\eta} - \frac{4}{45}\eta^3 + \frac{152}{2835}\eta^5 + O(\eta^7). \end{aligned} \tag{5.37}$$

For the region $\nu \in]0, \infty[$ we follow the numerical results of [64].
Naturally, beyond the scaling (one-loop) correction (5.34) we expect some systematic higher-loop expansion of the effective central charge (5.31). To evaluate these terms one has to take into account the higher-loop corrections to the RG evolution equation together with possible higher-loop modifications of the zero-mode dynamics (5.29).

5.2.4 TBA of the SST.

For general λ or rational λ the problem to find the Y-system is more complicated. We prefer to continue our analysis with $\lambda = 1/N$, where $N \in \mathbb{N}$. We remember that, with this strategy, we remain in the repulsive region $\lambda \in [0, 1/2[$. We have already explained that it is possible to explore the $\lambda \neq 1/N$ region by analytic continuation of this result. However, so far it is not clear if the sausage model is well defined for $\lambda \geq 1/2$. Investigations in this direction are in progress [80] and [81].

We write the SST_λ S-matrix in a more suitable form, in order to obtain a similar relation to (4.147). In fact, this system is equivalent to the spin 1-XXZ model, with

$$S(\theta) = F(\theta)R(\theta), \tag{5.38}$$

where $R(\theta)$ the $U_q(SU(2))$ R-matrix in spin 1 representation, i.e.

$$R(\theta) = \begin{pmatrix} a(\theta) & & & & & & & & \\ & b(\theta) & c(\theta) & & & & & & \\ & c(\theta) & b(\theta) & & & & & & \\ & & & d(\theta) & f(\theta) & g & & & \\ & & & f(\theta) & e(\theta) & f(\theta) & & & \\ & & & g & f(\theta) & d(\theta) & & & \\ & & & & & & b(\theta) & c(\theta) & \\ & & & & & & c(\theta) & b(\theta) & \\ & & & & & & & & a(\theta) \end{pmatrix} \tag{5.39}$$

and

$$F(\theta) = \frac{1}{\sinh \lambda(\theta + i\pi) \sinh \lambda(\theta - 2i\pi)}. \tag{5.40}$$

From (5.15) we can check that ($\lambda \equiv 1/N$, N being the number of magnons referred to $\hat{D}_N$ Dynkin diagram)

$$\begin{aligned}
&a(\theta) = \sinh[\lambda(\theta - i\pi)] \sinh[\lambda(\theta - 2i\pi)],\\
&b(\theta) = \sinh(\lambda\theta) \sinh[\lambda(\theta - i\pi)],\\
&c(\theta) = \sinh(-2i\pi\lambda) \sinh[\lambda(\theta - i\pi)],\\
&d(\theta) = \sinh[\lambda(\theta + i\pi)] \sinh(\lambda\theta),\\
&f(\theta) = \sinh(2i\pi\lambda) \sinh(\lambda\theta),\\
&g = \sinh(i\pi\lambda) \sinh(2i\pi\lambda),\\
&e(\theta) = \sinh(\lambda\theta) \sinh[\lambda(\theta - i\pi)] + \sinh(i\pi\lambda) \sinh(2i\pi\lambda).
\end{aligned} \tag{5.41}$$

Finally, we rewrite the complete S-matrix, in order to specify the order of factorization

$$S(\theta) = \begin{pmatrix}
\binom{++}{++} & & & & & & & & \\
& \binom{+0}{+0} & \binom{+0}{0+} & & & & & & \\
& \binom{0+}{+0} & \binom{0+}{0+} & & & & & & \\
& & & \binom{+-}{+-} & \binom{+-}{0\,0} & \binom{+-}{-+} & & & \\
& & & \binom{0\,0}{+-} & \binom{0\,0}{0\,0} & \binom{0\,0}{-+} & & & \\
& & & \binom{-+}{+-} & \binom{-+}{0\,0} & \binom{-+}{-+} & & & \\
& & & & & & \binom{0-}{0-} & \binom{0-}{-0} & \\
& & & & & & \binom{-0}{0-} & \binom{-0}{-0} & \\
& & & & & & & & \binom{--}{--}
\end{pmatrix} \tag{5.42}$$

Bethe equations.

The operatorial BA is

$$\begin{aligned}
e^{-ip(\theta_i)l} &= \mathrm{Tr}_i \prod_{j=1\neq i}^{N+1} S(\theta_{ij}) = \prod_{j=1\neq i}^{N+1} F(\theta_{ij}) \mathrm{Tr}_i \prod_{k=1\neq i}^{N+1} R_{ik}(\theta_{ik}) =\\
&= \prod_{j=1\neq i}^{N+1} F(\theta_{ij})\ t_2(\theta_i|\theta).
\end{aligned} \tag{5.43}$$

R-matrix can be decomposed in the sum of the *projectors* P^k on the states of the Hilbert space onto the π_k irreducible representation of $U_q(SU(2))$

$$R(\theta) = P^0 + \sum_{k=1}^{2s} \prod_{l=1}^{k} \frac{\sinh[\lambda(\theta + il\pi/2)]}{\sinh[\lambda(\theta - il\pi/2)]} P^l. \tag{5.44}$$

with $s = 1$. Eigenvalues of $t_2(\theta_i|\theta)$ are

$$t_2(x_0|x, \mathbf{u}) = I_1(x_0) + I_2(x_0) + I_3(x_0), \tag{5.45}$$

with the following expressions[8]

$$\begin{aligned}
I_1(x) &= \psi_0\left(x - i\frac{\mu}{2}\right)\psi_0\left(x - 3i\frac{\mu}{2}\right)\frac{Q_0\left(x + 3i\frac{\mu}{2}\right)}{Q_0\left(x - i\frac{\mu}{2}\right)}, \\
I_2(x) &= \psi_0\left(x - i\frac{\mu}{2}\right)\psi_0\left(x + i\frac{\mu}{2}\right)\frac{Q_0\left(x + 3i\frac{\mu}{2}\right)}{Q_0\left(x + i\frac{\mu}{2}\right)}\frac{Q_0\left(x - 3i\frac{\mu}{2}\right)}{Q_0\left(x - i\frac{\mu}{2}\right)}, \\
I_3(x) &= \psi_0\left(x - 3i\frac{\mu}{2}\right)\psi_0\left(x + i\frac{\mu}{2}\right)\frac{Q_0\left(x - 3i\frac{\mu}{2}\right)}{Q_0\left(x + i\frac{\mu}{2}\right)}.
\end{aligned} \tag{5.46}$$

We have defined $x \equiv \lambda\theta$, $\mu \equiv \lambda\pi$, ψ_0 and Q_0 as in (4.164), section 4.6.1. As we have made in the previous chapter, we write the complete system of equations

$$\begin{cases}
e^{-ip(x_t)l} = \displaystyle\prod_{j=1\neq t}^{N} F(x_{tj})\big(I_1(x_t) + I_2(x_t) + I_3(x_t)\big), \\
\dfrac{\psi_0(u_l + 2g)}{\psi_0(u_l - 2g)} = -\dfrac{Q_0(u_l + 2g)}{Q_0(u_l - 2g)} \qquad \text{(BAE)},
\end{cases} \tag{5.47}$$

where $g = i\mu/2$. The BAE previously evaluated for the SG model are quite different from BAE in (5.47). Here, we have a $2g$ shift in Bethe roots in both sides of the equation, while in (4.165) we have a g-shift for Ψ_0 and a $2g$ shift for Q_0.

Density of roots and TBA.

Evaluating, in the same way as in section 4.6.2, the density of roots from (5.47), we find for each λ the TBA for the SST_λ family

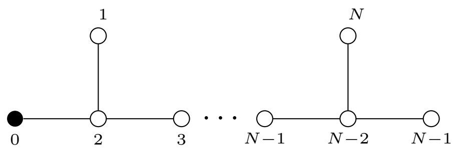
$$\begin{cases}
\delta_a^0 mR\cosh\theta = \epsilon_a(\theta) + \dfrac{1}{2\pi}\displaystyle\sum_{b=1}^{n} G_{ab}\left(\phi * L_b\right)(\theta), \\
\phi = \dfrac{1}{\cosh\theta},
\end{cases} \tag{5.48}$$

In this case, it turns out that the adjacency matrix G_{ab} represents the Dynkin diagram $\hat{D}_N$, i.e. the *affine Lie Algebra* D_N. The mass of the particle is on

[8] These new variables are dedicated to the Great Irene.

the fork, in the 0 node, as drawn down here

SST_λ



The finite-size ground state energy is (we call ρ the density of states instead of ν)

$$E(r) = -\frac{m}{2\pi} \int d\theta \cosh\theta \ln\left(1 + e^{\epsilon_0(\theta)}\right). \tag{5.49}$$

From the analysis made in section 4.3.2, the UV "central charge" becomes

$$c_{UV} = 2, \tag{5.50}$$

as we aspected. If we define, like in (5.31),

$$e(r, N) = 2 - c(r, N), \tag{5.51}$$

it turns out, from a accurate study of the behavior, at fixed N, of the Roger's Dilogarithm, that

$$e(\tau, N) \backsimeq \frac{3\pi^2(N-2)}{2\tau^2} + O(\frac{1}{\tau^3}), \tag{5.52}$$

where $\tau = -\log mr$. We explicit here that if $\tau \to \infty$ we are going into the UV limit.

From section 4.5 we have found very interesting relations (4.183)-(4.187) between e_{PF} and τ. Now, we *define*

$$\eta' = \frac{\tau}{N}, \tag{5.53}$$

If

$$\begin{cases} \tau \to \infty \\ N \to \infty \\ \eta' = \text{const} \end{cases} \tag{5.54}$$

then we are going into the UV limit of the $SST_{\lambda\to 0}$, so we are going into the $O(3)$ UV S-matrix theory.

On the other hand, we consider the "old" $\eta = \nu(t_0 - t)/4\pi$. In this case, if

$$\begin{cases} t \to -\infty \\ \nu \to 0 \\ \eta = \text{const} \end{cases} \tag{5.55}$$

then we are going into the UV limit of the $SSM_{\nu\to 0}$ theory, so we are going into the $O(3)$ UV S-matrix theory (the same as in the previous limit).
Now we can see that (5.52) can be rewritten as

$$\begin{aligned} e(\tau) &= \frac{3\pi^2(N-2)}{2\tau^2} + O(1/\tau^3) = \\ &= \frac{3\pi^2}{2}\frac{N^2}{\tau^2}\frac{1}{N} + O(\log\tau/\tau^2), \end{aligned} \tag{5.56}$$

We have just seen that

$$\lim_{N\to\infty} e(\eta') = \lim_{n\to\infty} e_{PF}(\eta_{|\nu=0}, n). \tag{5.57}$$

We want to demonstrate that $\lim_{n\to\infty} e_{PF}(r,n) = \lim_{\nu\to 0} e_\nu(\eta)$. But, from (5.56) we have found that

$$\begin{aligned} \lim_{n\to\infty} e_{PF}(r,n) &= \lim_{N\to\infty} \frac{3\pi^2}{2}\frac{N^2}{\tau^2}\frac{1}{N} + O(\log\tau/\tau^2) = \\ &= k(\eta')/N + O(\log\tau/\tau^2) \sim k(\eta')\lambda. \end{aligned} \tag{5.58}$$

It is possible, due to the symmetry of (5.24) with respect to axial rotations

$$X(x) \;\to\; X(x) + \xi \qquad \xi \in [0, 2\pi[\;\;, \tag{5.59}$$

to couple to the action an external gauge field A_μ

$$A_{SSM_\nu} = \int d^2x \frac{(\partial_\mu Y)^2 + (\partial_\mu X - iA_\mu)^2}{a(t) + b(t)\cosh(2Y)}. \tag{5.60}$$

Analyzing this action we can obtain the important 1-loop relation

$$\nu = 4\pi\lambda + O(\lambda^2). \tag{5.61}$$

Thanks to this relation, we can find that

$$\lim_{n\to\infty} e_{PF}(r,n) = k(\eta')\lambda + O(\lambda^2) = \frac{\nu k(\eta')}{4\pi} + O(\lambda^2), \tag{5.62}$$

i.e., the same relation of $e_\nu(\eta)$.

An impressive agreement was observed between the behavior of $k(\eta)$ in (5.34) (from (5.36)) and $k(\eta')$ in (5.62). We plot in Fig.(5.5) the same numerical results of [64]. This means that SSM_ν is the field theory of SST_λ.

5.2.5 Conclusions.

We have seen very important evidences of the fact that the SSM_ν field theory is the integrable deformation of the $O(3)$ NLSM. They coincide in the $\nu = 0$ point, they correspond point to point to the same FST family (namely, the SST_λ family) and they have the same UV behavior. It is sure to argue that the 1-loop approximation at the RG flow in (5.16) is the common trajectory of SSM_ν QFTs, for $\nu \in [0, 1]$ and with Φ like in (5.18).
In other words, we have found a family of trajectories dependent by ν, linked together by the correspondence with SST_λ and the same UV behavior.
Nevertheless, everything so far is an hypothesis without a solid proof. We would need the RG flow solutions to all higher orders of approximation.

Another problem is related to the two definitions of the sausage sigma model, that is the two ones gave as SST_λ and SSM_ν. In fact, we would like to know what exactly is ν and what exactly is its relation with λ, not only their 1-loop relation.

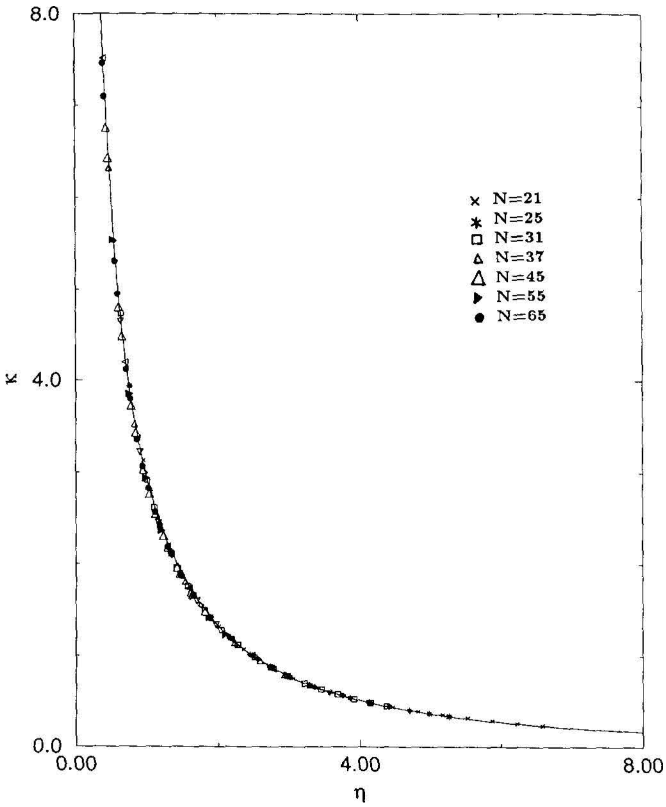


Figure 5.5: The lowest eigenvalue of (5.36), i.e. $k(\eta)$, is plotted (solid line) with respect to many numerical solutions of the TBA equations at different N (different point-shape). This image is taken from [64].

Chapter 6

Non Linear Integral Equations. A general method of resummation.

This chapter is entirely devoted to the general method of resummation of the infinite Y-system for the sine-Gordon model and for the Sausage Sigma Model into a *finite* set of equations named *Non Linear Integral Equations* or NLIEs. The method, in principle, is extendible to every Y-system, because it is based on a very general system, called *T-system*, which relates every transfer matrix of a particular representation to those of representations immediately next and previous.
In literature the most known method to resum infinite Y-system in a non-linear integral equation system is the so-called *Destri-De Vega* method or, for short, DDV [79]. DDV equation for the SG model is well known and with our resummation method we re-derive the same result. Instead, DDV for the SSM is not yet known. There exist only the NLIEs for the limit $N \to \infty$, i.e. for the $O(3)$ NLSM [82]. Here we deduce for the first time the NLIE for general $\lambda = 1/N, N \in \mathbb{N}$.

In section 6.1 we introduce T-system and TQ relations and we outline the details on the construction of the NLIE.
After that, in sections 6.2 and 6.3, we find NLIE for SG model and SSM, respectively.

6.1 Building NLIEs.

In this section we derive the general method to "resum" the infinite Y-system into a finite set of integral equations. We follow the notation used in [83], as we have made in section 4.5. In the following, $t_k^{(l)}$ indicates that the l is the $\mathfrak{su}(2)$ representation of the quantum space $\mathscr{H} = \left(\mathbb{C}^{l+1}\right)^{\otimes N}$ and k is the representation of the auxiliary space $\mathscr{A} = \mathbb{C}^{k+1}$. We focus our attention on the case $\mathfrak{su}(2)$ because the group of invariance for the SG and for the SSM is $U_q(SU(2))$.

T-system and TQ-relations

T-system is a fundamental structure which reflects the symmetries of the system taken in consideration in a quite direct way. We shall see that there exists a fundamental link between T-systems and Y-systems, which is the key observation of our analyses. We shall be able to write a particular set of non-linear integral equations which will be a *simple and finite version* of the universal (in some case infinite) Y-system.

The transfer matrix $t_k^{(l)}(x)$ is a solution of the celebrated T-*system* for simply laced Lie algebra $\mathfrak{su}(2)$. For a chain of spin $l/2$ the T-system is

$$t_k^{(l)^-} t_k^{(l)^+} = t_{k-1}^{(l)} t_{k+1}^{(l)} + \phi^{[k]} \overline{\phi}^{[-k]}. \tag{6.1}$$

$\phi(x)$ is a meromorphic function of x and

$$\begin{aligned} f^{\pm}(x) &= f(x \pm \frac{i\pi}{2}), \\ f^{[n]}(x) &= f(x + in\frac{\mu}{2}), \\ \overline{f}(x) &= f^*(x), \end{aligned} \tag{6.2}$$

where $\mu \in \mathbb{R}$ and $n \in \mathbb{Z}$. The T-system is consistently truncated at $t_{-1}(x) = 0$. Matching results found in [76], where the T-system for the $t_k^{(l)}$ transfer matrix of the $SU(2)$ or $U_q(SU(2))$ invariant problem is derived, we can write (6.1) with $\phi^{[k]}\overline{\phi}^{[-k]} = 1, \ \forall k = 0, 1, \cdots$

$$t_k^{(l)^-} t_k^{(l)^+} = t_{k-1}^{(l)} t_{k+1}^{(l)} + 1. \tag{6.3}$$

It is always possible to relate an A_n Y-system to the T-system in (6.1) [75]. We stress that the spin-chain is an auxiliary problem. We solve the integrable $l/2$-spin-chain problem invariant with respect to G, where G is a generic group, in order to solve the scattering problem for a field theory invariants with respect to the same group G. If all particles in the theory have spin $l/2$ then, as we have seen previously, it is possible to solve the Bethe-Yang equations by the eigenvalues of the transfer matrix t_l.

The first step of our method consists of transforming the Y-system of the theory, related to a generic diagram, into an A_n Y-system. Then, taking the T-system related to the theory, we build a correspondence between each element of the Y-system with each element of the T-system. Later, we shall find a system of few NLIEs whose solutions are the solutions of the initial Y-system.
We shall obtain the NLIEs for the SG and the SSM. Each theory is $U_q(SU(2))$-invariant, but the representation of the particle spectra are, respectively, the first ($l = 1$, i.e. $s = 1/2$-representation) and the second ($l = 2$, i.e. $s = 1$-representation) of $U_q(SU(2))$. Hereafter we omit the upper index (l) and the T-system will be written simply

$$t_k^- t_k^+ = t_{k-1} t_{k+1} + 1. \tag{6.4}$$

The T-system (6.1) is integrable and admits a Lax pair representation through the auxiliary problem

$$\begin{cases} t_{k+1} Q^{[k]} - t_k^- Q^{[k+2]} = \phi^{[k]} \overline{Q}^{[-k-2]}, \\ t_{k-1} \overline{Q}^{[-k-2]} - t_k^- \overline{Q}^{[-k]} = -\overline{\phi}^{[-k]} Q^{[k]}, \end{cases} \tag{6.5}$$

where $\overline{Q}(x)$ and $\overline{\phi}(x)$ are, respectively, the hermitian conjugates of $Q(x)$ and $\phi(x)$. These are the so-called *TQ-relations* which link the k-transfer matrices with the function[1] $Q(x)$. From the linear system (6.5) one finds a solution for t_k in terms of Q and $\overline{Q}$ [84]

$$t_k(x) = \frac{Q^{[k+1]}}{Q^{[-k+1]}} t_0^{[-k]} + Q^{[k+1]} \overline{Q}^{[-k-1]} \sum_{j=1}^{k} \xi^{[-k+2j]}, \tag{6.6}$$

where

$$\xi(x) = \frac{\phi(x - i\frac{\mu}{2})}{Q(x + i\frac{\mu}{2}) Q(x - i\frac{\mu}{2})}. \tag{6.7}$$

We define

$$\begin{aligned} y_k(x) &= \frac{t_{k-1}(x) t_{k+1}(x)}{\phi(x + ik\frac{\mu}{2}) \overline{\phi}(x - ik\frac{\mu}{2})}, \\ Y_k(x) &= \frac{t_k(x + i\frac{\mu}{2}) t_k(x - i\frac{\mu}{2})}{\phi(x + ik\frac{\mu}{2}) \overline{\phi}(x - ik\frac{\mu}{2})} \end{aligned} \tag{6.8}$$

[1] $Q(x)$ is often called in the literature “Baxter Q-operator. The name “operator” is for historical reasons, but it is a function of x.

with $\mu = \pi$. With these definitions, the T-system (6.4) becomes simply

$$1 + y_k(x) = Y_k(x). \tag{6.9}$$

One can check that

$$\begin{aligned} &y_k\left(x + i\frac{\pi}{2}\right) y_k\left(x - i\frac{\pi}{2}\right) = Y_{k-1}(x) Y_{k+1}(x), \qquad k = 1, 2, \cdots \\ &y_0(x) = 0, \end{aligned} \tag{6.10}$$

that is the Y-system for the TBA equations related to an A_∞ Dynkin Diagram. The Y-system in (6.10), T-system and TQ-relations are gauge invariant

$$\begin{aligned} t_k(x) &\longrightarrow g^{[k]}\overline{g}^{[-k]} t_k(x), \\ \phi(x) &\longrightarrow g^- g^+ \phi(x), \\ \overline{\phi}(x) &\longrightarrow \overline{g}^- \overline{g}^+ \overline{\phi}(x), \\ Q(x) &\longrightarrow g^- Q(x). \end{aligned} \tag{6.11}$$

Hereafter we consider the simplified definitions of the Y functions (thanks to [76])

$$\begin{aligned} y_k(x) &= t_{k-1}(x) t_{k+1}(x), \\ Y_k(x) &= t_k^+(x) t_k^-(x). \end{aligned} \tag{6.12}$$

We can fix the gauge invariance writing the function $\phi(x)$ as a product of functions $\psi_0(x)$ (see section 4.4.1), depending on the system. In fact, it is always possible to choose $Q(u_j) = 0$ and find, from (6.6), the Bethe Ansatz Equations of the problem. For instance,

$$\begin{aligned} \phi(x) &= \psi(x + i\frac{\mu}{2}) \qquad \text{for SG}, \\ \phi(x) &= \psi(x)\psi(x + i\mu) \qquad \text{sor SSM}, \end{aligned} \tag{6.13}$$

with the respective normalized variables x (see previous chapters). Some elucidations are in order here. Since, in a specific gauge (i.e. (6.13)), for $k = 1$, (6.6) is a polynomial of degree N or, for all $k \geq 2$, of degree $2N$, where N is the total number of the particles, $t_k(x)$ can have N or $2N$ roots. It was shown that in the physical strip[2] $0 <\mathrm{Im} x < \pi/p$, $t_k(x)$ has only real roots. Moreover, it is possible that the transfer matrix has no roots. In this case ([75], [78]) the TBA obtained from (6.8) refers to the ground state energy. Studying this problem including roots of the transfer matrix gives the TBA for excited states.

[2] p is the coupling parameter of the SG defined in section 4.6.

The functions b_k and B_k.

We introduce other two useful variables, $b_k(x)$ and $B_k(x)$, defined for $k \geq 0$ as

$$\begin{cases} b_k = \dfrac{Q^{[k+2]} t_k^-}{\overline{Q}^{[-k-2]}}; \\ B_k = \dfrac{Q^{[k]} t_{k+1}}{\overline{Q}^{[-k-2]}}. \end{cases} \tag{6.14}$$

It's easy to see from (6.12) that

$$\begin{aligned} B_k &= 1 + b_k; \\ b_k \bar{b}_k &= Y_k; \\ B_k^+ \overline{B}_k^- &= Y_{k+1}. \end{aligned} \tag{6.15}$$

TBA Dynkin Diagrams for SG and SSM have one massive node (A_1) and a "tower" of magnonic nodes (D_n and $\hat{D}_n$ respectively). For these two models we want to obtain a *reduced* Dynkin Diagram like that in Fig.(6.3). We illustrate here a method that maps the Dynkin diagrams mentioned above for the SG and the SSM into an A_l one.

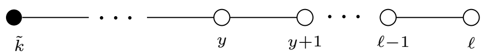


Figure 6.3: This is our *final* A_ℓ Dynkin diagram. We have denoted by $\tilde{k}$ the first node, which is massive (for this reason we paint it black). In both cases analyzed, $l = n - 1$.

6.2 Ground state NLIE for the SG theory.

We show how to obtain the NLIE from the SG Y-system (i.e., from its S-matrix). It will turn out to be equal to the DDV non-linear integral equation. This fact can be used to check our method.

We start from a D_{N+1} Dynkin diagram. We construct the reduced Y-system from the original one

$$\begin{cases} y_1^+ y_1^- = Y_2, \\ y_n^+ y_n^- = Y_{n-1} Y_{n+1}, \quad 2 \leq n \leq N-2, \\ y_{N-1}^+ y_{N-1}^- = Y_{N-1} Y_N Y_{N+1}, \\ y_N^+ y_N^- = y_{N+1}^+ y_{N+1}^- = Y_{N-1} \end{cases} \tag{6.16}$$

via some substitution

$$\begin{cases} Y_n = Z_n, \quad 1 \leq n \leq N-1, \\ Y_{N+1} Y_N = Z_N. \end{cases} \tag{6.17}$$

Identifying Z_k and $z_k \equiv Z_k - 1$ as in (6.12), we obtain the T-system (6.4). Now, because of the equality $y_n^+ y_N^- = Y_{N-1} = y_{n+1}^+ y_{N+1}^-$, we easily find that $y_N = y_{N+1}$, that is $Y_{N-1} = t_{N-1}^+ t_{N-1}^- = y_n^+ y_N^- = y_{n+1}^+ y_{N+1}^-$, so

$$t_{N-1} = y_N = y_{N+1} \tag{6.18}$$

and, because of

$$\begin{cases} Z_N = t_N^+ t_N^- = 1 + t_{N+1} t_{N-1}, \\ Z_N = Y_N Y_{N+1} = (1 + t_{N-1})^2, \end{cases} \tag{6.19}$$

for N even and odd, we obtain that

$$t_{N+1} = 2 + t_{N-1}. \tag{6.20}$$

This truncation has an important consequence. From the TQ-system, one finds that

$$\begin{aligned} t_{k-1}^{[k]} \left(\overline{Q}^{++} + \overline{Q}^{--} \right) &= \overline{Q} \left(t_k^{[k-1]} + t_{k-2}^{[k+1]} \right), \\ Q \left(t_k^{[-k+1]} + t_{k-2}^{[-k-1]} \right) &= t_{k-1}^{[-k]} \left(Q^{++} + Q^{--} \right), \end{aligned} \tag{6.21}$$

Calling

$$A \equiv \frac{Q^{++} + Q^{--}}{Q}, \qquad A_k = \frac{t_k^{[-k+1]} + t_{k-2}^{[-k-1]}}{t_{k-1}^{[-k]}} \tag{6.22}$$

(and the respective hermitian conjugates) we note from (6.21) that $A = A_k$, i.e. it doesn't depend on the index k. For $k = N+1$ we obtain

$$Q^{[2N+2]} = \overline{Q}. \tag{6.23}$$

We prefer from now to work in Logarithmic Fourier space, that is[3]

$$\begin{cases} \hat{f}(w) = \displaystyle\int \frac{dx}{2\pi} e^{iwx} \log f(x), \\ \displaystyle\int \frac{dx}{2\pi} e^{iwx} \log f(x + ia\mu) = p^a \hat{f}(w), \qquad p = e^{w\mu}. \end{cases} \tag{6.24}$$

[3]Hereafter, the integrals are supposed to go from $-\infty$ to ∞ if not otherwise stated.

Taking (6.14), we obtain

$$\begin{aligned}\hat{b}_k &= p^{k+2}\hat{Q} + p^{-1}\hat{t}_k - p^{-k-2}\hat{\overline{Q}},\\ \hat{\overline{b}}_k &= p^{-k-2}\hat{\overline{Q}} + p\hat{t}_k - p^{k+2}\hat{Q}.\end{aligned} \tag{6.25}$$

In the same way, we can find the right Log-Fourier values for B_k and $\overline{B}_k$. Inserting (6.23)

$$\begin{aligned}\hat{b}_k &= \hat{Q}\left(p^{k+2} - p^{2N-k}\right) + p^{-1}\hat{t}_k,\\ \hat{\overline{b}}_k &= \hat{Q}\left(p^{2N-k} - p^{k+2}\right) + p\ \hat{t}_k,\\ \hat{B}_k &= \hat{Q}\left(p^{k} - p^{2N-k}\right) + \hat{t}_{k+1},\\ \hat{\overline{B}}_k &= \hat{Q}\left(p^{2N-k+2} - p^{k+2}\right) + \hat{t}_{k+1},\end{aligned} \tag{6.26}$$

Then it turns out that

$$\hat{Q} = \frac{\left(\hat{B}_k - \hat{\overline{B}}_k\right)}{(1+p^2)(p^k - p^{2N-k})}. \tag{6.27}$$

We obtain

$$\begin{aligned}\hat{b}_k &= \tilde{K}\left(\hat{B}_k - \hat{\overline{B}}_k\right) + \hat{t}_k p^{-1},\\ \hat{\overline{b}}_k &= -\tilde{K}\left(\hat{B}_k - \hat{\overline{B}}_k\right) + \hat{t}_k p,\end{aligned} \tag{6.28}$$

where

$$\begin{cases}\tilde{K} = \dfrac{p^{N-k-1} - p^{k-N+1}}{p^{N-k} - p^{k-N}}\tilde{s},\\ \tilde{s} = \dfrac{1}{p+p^{-1}},\end{cases} \tag{6.29}$$

We can make the *resummation*, that is, we consider only $k = 1$, which means, graphically, that all nodes are put together into a single node (the first, Fig(6.4)).

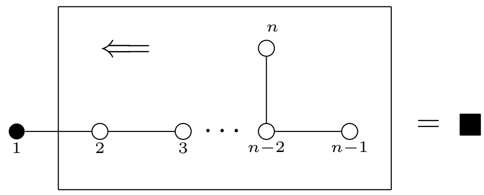


Figure 4.4: a graphic representation of the resummation of the TBA Dynkin diagram for the SG. The full-filled black box indicates that the massive term is included in the first NLIE (see (6.33))

We obtain

$$\begin{cases} \hat{b}_1 = \tilde{K}_1 \left(\hat{B}_1 - \hat{\overline{B}}_1 \right) + \hat{t}_1 p^{-1}, \\ \hat{\overline{b}}_1 = -\tilde{K}_1 \left(\hat{B}_1 - \hat{\overline{B}}_1 \right) + \hat{t}_1 p, \end{cases} \tag{6.30}$$

where $\tilde{K}_1 \equiv \tilde{K}_{|k=1}$. To anti-Fourier transform this back to the x space, we have to observe that the function b_1 has zeros on the real axis, therefore $\ln b_1$ has singularities that should be avoided. So, we define a shifted function $\hat{a} = p^{\epsilon}\hat{b}_1$, where $\epsilon \to 0$, $\epsilon > 0$. For the same reason $\hat{\overline{a}} = p^{-\epsilon}\hat{\overline{b}}_1$, introducing also $A = 1 + a$ and $\overline{A} = 1 + \overline{a}$. Finally, we can write the NLIE

$$\hat{a} = \tilde{K}_1 \hat{A} - \tilde{K}_1 p^{2\epsilon} \hat{\overline{A}} + \tilde{s}\hat{Y}_1 p^{-1+\epsilon} \tag{6.31}$$

We see that it is possible to use only one equation in (6.30) (we chose the first). We note that this extreme reduction has been possible thanks to the equivalence $Y_1 = t_1^+ t_1^-$ (see (6.17)).
Writing $\tilde{K}_1$ as

$$\tilde{K}_1 = \frac{\sinh\left[(N-2)\frac{w\pi}{2}\right]}{2\sinh\left[(N-1)\frac{w\pi}{2}\right]\cosh\left[\frac{w\pi}{2}\right]}, \tag{6.32}$$

we can derive the so-called NLIE equation for the SG in the configuration space from (6.31) by integrating over w. The variable θ, that is the rapidity of the physical particle, is related to the parameter x by $x = \theta/p$. The final result is:

$$\begin{aligned} \ln\left(a(\theta)\right) = {}& iml\sinh(\theta) + \\ & + G * \ln(1+a)(\theta + i\epsilon) + \\ & - G * \ln(1+\overline{a})(\theta - i\epsilon), \end{aligned} \tag{6.33}$$

where we have defined the *kernel* $G(\theta)$ as

$$G(\theta) = \int dk \left\{ e^{-ik\theta} \frac{\sinh\left[(N-2)\frac{k\pi}{2}\right]}{2\sinh\left[(N-1)\frac{k\pi}{2}\right]\cosh\left[\frac{k\pi}{2}\right]} \right\} \tag{6.34}$$

and the $\ln(x)$ is the fundamental logarithmic function with the branch cut on the real negative axis.
We have found the exact result we would have obtained, i.e. the same result as in [78], [79] and [73], if we identify their function $Z(\theta)$ as

$$iZ(\theta) = \ln a(\theta), \tag{6.35}$$

where $Z(\theta)$ is the *counting function* defined as $F(\theta) = \exp\left(iZ(\theta)\right)$, where

$$F(\theta) = \frac{Q(\theta/p + i\pi/p)\psi_0(\theta/p - i\pi/(2p))}{Q(\theta/p - i\pi/p)\psi_0(\theta/p + i\pi/(2p))}, \tag{6.36}$$

The D_N Dynkin diagram is associated with the SG TBA at the points

$$N - 1 = p, \tag{6.37}$$

where $p = \frac{\beta^2}{8\pi - \beta^2}$ as usual.
This coincides with results found by Fateev in [61] (see the note at pag. 101). We see that (6.34) has an essential singularity in $(N = 1) \equiv (p = 0)$. This is the known SG singularity at $\beta^2 = 0$. In $N = 2$ we see a zero of the function $G(\theta)$, which means that $\log a(\theta) = iml \sinh(\theta)$, i.e. the theory is free. This coincides with the known SG free point $\beta^2 = 4\pi$ where it reduces to a free fermion. Otherwise, when $p \to \infty$, $N \to \infty$, we go towards $\beta^2 = 8\pi$, i.e. the $SU(2)$ symmetric SG theory, also known, in fermionic language, as $SU(2)$ Gross-Neveu model. The kernel $G(\theta)$ in this case becomes [73]

$$G(\theta) = \int dk e^{-ik\theta} \frac{e^{-\frac{|k|\pi}{2}}}{2\cosh\left[\frac{k\pi}{2}\right]}. \tag{6.38}$$

Naturally this analyses is been made with $p \in \mathbb{N}$, and this means that we haven't encoded all values of β^2. The analyses of the most general case (p rational or even irrational) becomes more cumbersome and we omit it in this discussion, but it can be done by resorting to techniques of continued fraction decomposition, introduced for the XXZ spin-chain by Takahashi and Suzuki [85] and exploited for the SG TBA by Tateo [86].

6.3 Ground state NLIEs for the SSM theory.

The Dynkin diagram associated to the TBA SST_λ is $\hat{D}_N$, as we have seen in section 5.2.4

(Our SST_λ)

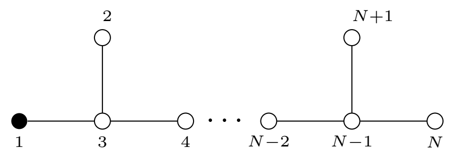


Figure 6.5: The Dynkin diagram associated with SST_λ with new labels.

In order to obtain A_N as in Fig.(6.3), we take

$$\begin{cases} y_1^+ y_1^- = y_2^+ y_2^- = Y_3, \\ y_3^+ y_3^- = Y_1 Y_2 Y_4, \\ y_n^+ y_n^- = Y_{n-1} Y_{n+1}, \quad 4 \le n \le N-2, \\ y_{N-1}^+ y_{N-1}^- = Y_{N-2} Y_N Y_{N+1}, \\ y_N^+ y_N^- = y_{N+1}^+ y_{N+1}^- = Y_{N-1}. \end{cases} \tag{6.39}$$

Making the following substitution (from $y_k \equiv z_k$)

$$\begin{cases} Y_1 Y_2 = Z_2, \\ Y_n = Z_n \quad 3 \leq n \leq N-1, \\ Y_N Y_{N+1} = Z_N, \end{cases} \tag{6.40}$$

we obtain again

$$t_{N+1} = 2 + t_{N-1}. \tag{6.41}$$

If we look at (6.20), we observe that the two expressions are equivalent. In fact, both D_n that $\hat{D}_n$ have the same "fork" in the end. The only difference between them is the massive tail, i.e. the first part of the diagram. We shall see how this fact influences the evaluation of NLIEs for the SSM.

From the TQ-relation (6.5) we obtain the same result as in (6.23)-(6.29). In order to resum, we observe that here we have $Z_2 = Y_1 Y_2$, that is

$$\hat{Z}_2 = \hat{Y}_1 + \hat{Y}_2, \tag{6.42}$$

which means that we have two unknowns, other then a (or $\overline{a}$). From (4.107) we can check that

$$y_1(x) = e^{-mR\cosh(x)} y_2(x) \tag{6.43}$$

So it is possible to have a closed system if we take $k = 2$ in (6.28). This gives the NLIE for the SSM

$$\begin{cases} \hat{a}_2 = \tilde{K}_2 \hat{A}_2 - \tilde{K}_2 p^{2\epsilon} \hat{\overline{A}}_2 + \tilde{s}(\hat{Y}_2 + \hat{Y}_1) p^{\epsilon-1} \\ \hat{y}_2 = p^{1-\epsilon} \tilde{s} \hat{A}_2 + p^{\epsilon-1} \tilde{s} \hat{\overline{A}}_2 \end{cases} \tag{6.44}$$

We could sketch our method in Fig.(6.6)

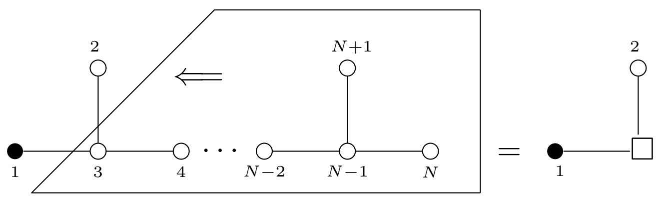


Figure 6.5: a graphic representation of the resummation of the TBA Dynkin diagram for the SSM. The empty box indicates that the massive term is not included in the first NLIE (see (6.46)) Checking that

$$\tilde{K}_{|k=2} \equiv \tilde{K}_2 = \frac{\sinh[(N-3)\frac{w\pi}{2}]}{2\sinh[(N-2)\frac{w\pi}{2}]\cosh[\frac{w\pi}{2}]}, \tag{6.45}$$

we can derive from (6.44), by integration over w, the NLIEs in the configuration space

$$\begin{cases} \log a_2(\theta) = G * \ln(1+a_2)\left(\theta + i\epsilon\frac{\pi}{2}\right) - G * \ln(1+\overline{a}_2)\left(\theta - i\epsilon\frac{\pi}{2}\right) + \\ \qquad\qquad + K^{[-\pi/2]} * \ln(Y_1 \cdot Y_2)\Big(\theta\Big), \\ \ln y_2(\theta) = K^{[\pi/2]} * \ln(1+a_2)\left(\theta + i\epsilon\frac{\pi}{2}\right) + K^{[-\pi/2]} * \ln(1+\overline{a}_2)\left(\theta - i\epsilon\frac{\pi}{2}\right), \\ \ln y_1(\theta) = -ml\cosh\theta + \ln y_2(\theta). \end{cases} \tag{6.46}$$

where

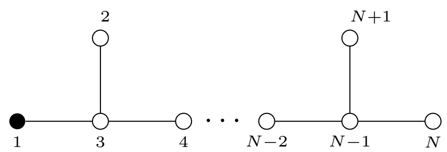
$$\begin{aligned} G(\theta) &= \int dw \left\{ e^{-iw\theta} \frac{\sinh\left[(N-3)\frac{w\pi}{2}\right]}{2\sinh\left[(N-2)\frac{w\pi}{2}\right]\cosh\left[\frac{w\pi}{2}\right]} \right\}, \\ K(\theta) &= \frac{1}{2\pi\cosh\theta}. \end{aligned} \tag{6.47}$$

We have restored the "old" variable θ, that is the rapidity of the physical particle.

How much is λ?

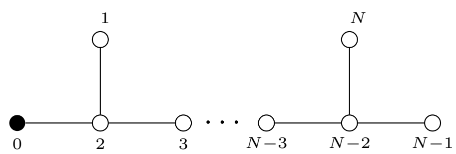
The kernel $G(\theta)$ in (6.47) is obtained from the Dynkin diagram $\hat{D}_N$. The Dynkin diagram associated to SST_λ is $\hat{D}_N$.

(Our SST_λ)



From Fateev, Onofri, Zamolodchikov (FOZ) [64] we can take $\lambda = 1/N$. They use another way of labeling the nodes:

(FOZ SST_λ)



Of course, the physics must be independent from labeling and N in $G(\theta)$ is not a physical quantity but it is linked to our way of labeling $\hat{D}_N$. If we call $x \in \mathbb{N}$ a "FOZ label" and $\overline{x} \in \mathbb{N}$ "our label", we can check that

$$\overline{x} = x + 1. \tag{6.48}$$

For instance, where $\overline{x} = 1$, $x = 0$ and so $1 = 0 + 1$. Note now that "FOZ" $\lambda = 1/N$ is a physical relation. So, if we want to relate (6.47) with λ, first we have to translate N in $N+1$ (in fact we must write $G(\theta) = f(\theta, \overline{N})$). We obtain the same kernel as in SG (6.34), but with a different relation to the physics of the model:

$$G(\theta) = \int dw \left\{ e^{-iw\theta} \frac{\sinh\left[(N-2)\frac{w\pi}{2}\right]}{2\sinh\left[(N-1)\frac{w\pi}{2}\right]\cosh\left[\frac{w\pi}{2}\right]} \right\}, \tag{6.49}$$

Now, if $\lambda = 1/2$ then $G(\theta) = 0$ and from (6.46) we see that the theory is free, since the term $a_2(\theta)$ depends only on the mass m. If $\lambda = 1$ then $N = 1$ and G encounters an essential singularity. If we take the limit $N \to \infty$ ($\lambda \to 0$) we get the same result of [75].

Moreover, we notice an interesting fact. From (6.37), i.e. $p = N - 1$ in SG, we obtain the relation

$$p = \frac{1}{\lambda} - 1. \tag{6.50}$$

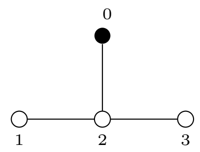
It is easy to see from (6.50) that

$$\begin{cases} p = 0 & \lambda = 1 & \text{essential singularities coincide;} \\ p = 1 & \lambda = 1/2 & \text{free theory;} \\ p \to \infty & \lambda \to 0 & \text{completely repulsive system.} \end{cases} \tag{6.51}$$

The third relation in (6.51) shows another interesting duality between these two families. In fact in the $\lambda \to 0$ regime we obtain the $SU(2)$-invariant $O(3)$ NLSM. If we perform the $p \to \infty$ limit for the SG, we obtain the $SU(2)$-invariant Gross-Neveu model.

We notice that in the $\lambda \to 0$ limit we obtain a TBA referred to D_∞, that was identified as the TBA of the $O(3)$ NLSM by Zamolodchikov and Zamolodchikov [61].

(6.50) can be checked looking at the Parafermion series H_N. In $N = 4$ we have the Dynkin diagram:



This model has $c = 1$ from $c = 2(N-1)/N+2$, as SG. In fact, H_4 is SG in $\beta^2 = 6\pi$ or $p = 3$. Now, if we add a magnon in the right way, we can build $\hat{D}_4$, i.e. the sausage model in $\lambda = 1/4$. We find that the relation (6.39) holds. It is interesting to note that the SSM and SG don't coincide in general, but parafermions and SG do (in $N = 4$).

6.4 Conclusions and outlooks.

What we have done in this thesis is the development of an original method to solve the many-variables (sometimes infinite) Y-system for the Sine-Gordon (SG) and for the Sausage Sigma Model (SSM). This method is based upon general relations, the so-called T-system, a semi-infinite integrable system whose elements can be put in a one-to-one relation with the elements of a generic Y-system. A solution of the T-system is the transfer matrix, whose eigenvalues constraint (quantize) the spectrum of the physical particle rapidities of the theory.
We have seen that the SG and SSM models are two different spin representations (respectively, the 1/2-th and the 1-th) of the same $U_q(SU(2))$ algebra. This method, thanks to the generality of the T-system, works not only for these two representations, but for any representation of any classic or quantum Lie Algebra [76].
We have turned out that this method is independent from the way of labeling the TBA Dynkin diagram and involves quite simple calculations. For this reasons it represents a strong tool in order to solve an integrable system of particles in 2 dimension.
This method can give more insight on different important and attractive fields of research. In the following, we give a list of some possible targets which we want to investigate in the next future.

We are actively studying [81] the Sausage S-matrix in the regime $\lambda > 1/2$. For these values there appear poles in the S-matrix for physical values of the rapidity θ. At these values, the scattering produces a bound state which mass is a function of the pole. We have found that these bound states organize in two multiplets, invariant with respect to $U(1)$, and have all the same mass. These fact increases the difficulty to close the S-matrix bootstrap, but should lead to very interesting results about analytic continuation of parameters like λ in integrable models.

- Adding a CDD factor to the SSM S-matrix defines a new class of models, the so called "Fateev models" [87]. These theories show very intriguing dualities between a sigma-model description and a more "traditional" Lagrangian with exponential potentials like the Affine Toda Field Theory ones. A model very close to the Fateev ones has been proposed recently by Basso and Rey [88] and plays a crucial role in the integrable sector of the AdS/CFT correspondence.

- The NLIEs proposed here describe completely the vacuum energy of the theory as a function of the size. Therefore, we can collect valuable informations on the Casimir effect; moreover, by using techniques first introduced by Lüscher, we can confirm the validity of the equation for any $\lambda \in]0, 1/2]$ (and not only for $\lambda = 1/N$ $N \in \mathbb{N}$) extracting information from the S-matrix.

- However, to claim to have a complete control over the energy spectrum, one should derive the NLIE for the excited states. This is possible specifying the analytic structure (zeros, poles, asymptotic behavior) of the $t_k(\theta)$ functions. This represents another possible outlook of this thesis.

- New realizations on gauge/string duality can be found by a deep investigation on the XXZ spin-chain for representations greater than 1. One can drive these studies analyzing quantum deformed NLSMs with $s > 1$.

- Finally, integrable generalization of the SSM for general $O(N)$ NLSM are invoked. The $O(4) = SU(2) \times SU(2)$ case is already known as the $S - S$ model [89], but the most interesting case, due to its importance in the AdS/CFT theory, is the $O(6)$ NLSM. Its deformations could suggest entire families of dual models where the very large $N = 4$ superconformal symmetry of the related gauge theory is broken towards more realistic models, which might have relevance in the GUT phenomena.

Appendix A

Very basic notions on groups and representations.

A *group* G is a set of elements embedded by an ordered product and satisfying:

- If f and g belong to G, then $h = fg \in G$.
- For $f, g, h \in G$, $f(gh) = (fg)h$.
- There exists $e \in G$ (the *identity element*) such that $\forall\, f \in G,\ fe = ef = f$.
- $\forall\, f \in G$, there exists $f^{-1} \in G$ such that $ff^{-1} = f^{-1}f = e$.

A *representation* D of the group G is a mapping between each elements of G onto a set of linear operators with the following properties

- ⋆ $D(e) = \mathbb{I}$ where $\mathbb{I}$ is the identity element of the space on which D acts.
- ⋆ $\forall\, g_1, g_2 \in G,\ D(g_1)D(g_2) = D(g_1 g_2)$.

If the elements of a group H belong also to the group G, we say that H is *subgroup* of G and $H \subset G$. We define *right-coset* of the subgroup H *in* the group G the set of elements of the form Hg, for some fixed $g \in G$. Similarly it is possible to define *left-coset*. Each coset can be seen as an element of a space, the so-called *coset-space*.
We call $\alpha(g)$ the *action* of the the group G, where $g \in G$. If $\mathscr{M}$ is a manifold, the action of G on $\mathscr{M}$ is $\alpha(g)m$ (or α_m), where $m \in \mathscr{M}$. We define the *stabilizer* of the map α_m the set

$$G_m = \{g \in G :\ \alpha(g)m = m\}. \tag{A.1}$$

The stabilizer G_m is a group.
An *homogeneous* space is a differentiable manifold $\mathscr{X}$ with a transitive action of G on it. Elements of g are called *symmetries* of $\mathscr{X}$.
If $\{X_a\}$ is a set of generators of a Lie Group ([21] and [22]), then the *adjoint representation* Ad is defined

$$[T_a, T_b] = i f_{abc} T_c \tag{A.2}$$

where

$$\begin{aligned} &[X_a, X_b] = i f_{abc} X_c, \\ &[T_a]_{bc} \equiv -i f_{abc}. \end{aligned} \tag{A.3}$$

If $\mathfrak{g}$ is the algebra of G homogeneous, G is *reductive*, that is there exists $\mathfrak{m}$ Ad_g-invariant such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{A.4}$$

where $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. In this case, the following relations hold

$$\begin{aligned} &[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \\ &[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \end{aligned} \tag{A.5}$$

and, if the $\mathfrak{g}$ is a *symmetric* space, then

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{A.6}$$

Appendix B

Factorization of the S-matrix in $1+1$ dimensions.

We give here the demonstration of the factorization of the S-matrix and the absence of particle production in some $1+1$ QFT, following the original paper [61]. A QFT with 2 or more conserved charges different from energy and momentum and with only massive particles in the spectrum has a factorized S-matrix and the set of initial momenta is conserved.
Consider such a theory, with only massive particles organized in mass multiplets, each particle in the multiplet having different internal symmetric charges. If $|p\rangle$ is an eigenstate of the momentum operator, then is also the eigenstate of the mass operator $M^2 \equiv P^\mu P_\mu$ such that $|p\rangle$ is a linear combination of the particle of the same mass multiplet.
We assume

1. The theory presents two conserved charges, Q^+ and Q^- which transform under the Lorentz group

$$Q'^+ = \Lambda^{+q} Q^+ \qquad Q'^- = \Lambda^{-n} Q^-, \tag{B.1}$$

   where q and n are odd integers satisfying $q \geq n > 1$. This ensures that the two charges aren't vectors or scalars. Note that, if $n = q$, the theory is parity invariant and it need only one conserved charge for the factorization.

2. $Q^\pm = \int dx j_0^\pm$ where $j_0^\pm$in the temporal component of a conserved local current. So $[Q^\pm, P^\mu] = 0$ and there exist a common set of eigenstates between Q and P.

3. $[Q^+, Q^-] = 0$.

4. Any non trivial linear combination between a multiplet particles can't be annihilated by $Q^{\pm}$.

As a consequence, there exist in each multiplet a set of single momentum states $|p_a\rangle$ (a labels the particle in the multiplet) which are eigenvalues of $Q^{\pm}$. Calling $\omega_a^{\pm}$ these eigenvalues, from (B.1) we obtain

$$\begin{aligned} Q^{+}|p_a\rangle &= \omega_a^{+}|p_a\rangle = \eta_a^{+}(p)^q|p_a\rangle; \\ Q^{-}|p_a\rangle &= \omega_a^{-}|p_a\rangle = \eta_a^{-}(p^{-})^n|p_a\rangle, \end{aligned} \tag{B.2}$$

where $\eta_a^{\pm}$ are Lorentz scalars depending on the particular combination of states.

The action of $Q^{\pm}$ on a widely separated multiparticle states, due to the local nature of the operators, is the sum of the single actions on each state

$$Q^{+}|p_1\cdots p_t\rangle = \left[\sum_{i=1}^{t}\eta_i^{+}(p_i)^q\right]|p_1\cdots p_t\rangle, \tag{B.3}$$

and similarly for Q^{-}. From (B.3) we note the asymptotic conservation of $\sum\eta_i^{+}(p_i)^q$ and $\sum\eta_i^{+}(p_i^{-})^n$.

Localized states.

A linear combination of operators $Q^{\pm}$ has the same properties of $Q^{\pm}$ except for the values of the eigenvalues. A useful combination will be

$$Q_\theta \equiv \frac{Q^{+}\cos\theta}{q} - \frac{Q^{-}\sin\theta}{n}. \tag{B.4}$$

A general scattering amplitude for a theory with conserved charges $Q^{\pm}$ is

$${}_{OUT}\langle p_1\cdots p_t|e^{-i\alpha Q_\theta}Se^{i\alpha Q_\theta}|q_1\cdots q_k\rangle_{IN} \tag{B.5}$$

with $\alpha\in\mathbb{R}$.

Suppose the states (B.5) be localized in space and, from $Q_{\theta\pm\pi} = -Q_\theta$, take $\alpha > 0$ and $0 \leq \theta < 2\pi$. The wave function of a localized single-particle state with mean momentum $(\overline{E}, \overline{p})$ is

$$\Psi(x,t) = N\int dp\; e^{f(p)}, \tag{B.6}$$

N being a normalization factor, and

$$\begin{aligned} f(p) &= -\frac{(p-\overline{p})^2}{2(\overline{E}\delta\Phi)^2} + i(p(x-x_0) - E(t-t_0)), \\ E &= (p^2+\mu^2)^{1/2} = \gamma\mu, \\ \gamma &= \left(\frac{p^2}{\mu^2}+1\right)^{1/2}, \end{aligned} \tag{B.7}$$

where μ is the mass of the particle and (t_0, x_0) represents the space-time point where the wave-packet spread is minimum. $\delta\Phi$ is half the velocity spread of the wave packet evaluated in the $\overline{p} = 0$ frame of reference. Therefore

$$0 < \delta\Phi \ll 1, \tag{B.8}$$

which avoids the problem to localize a single relativistic particle in a region of the order of the Compton wavelength $L_c = 2\pi\hbar$. By the stationary-wave method we find $\overline{x}(t)$, i.e. the center of the wave-packet at time t

$$\overline{x}(t) = x_0 t + \overline{v}(t - t_0), \tag{B.9}$$

with $\overline{v} = \overline{p}/\overline{E}$ (*mean velocity*). The width of the packet for large $|t - t_0|$ measures

$$\delta x(t) = k|t - t_0|(\delta\Phi/\overline{\gamma}^2) \tag{B.10}$$

If we ask the probability to find the particle outside $(\overline{x} + \delta x, \overline{x} - \delta x)$ to be small, $k \in \mathbb{R}$ will be small enough. In this way, only $\delta\Phi$ is the free parameter in the wave packet, but it must satisfy (B.8).
Acting with $\exp(i\alpha Q_\theta)$ on the (B.6) wave packet, one finds that the most likely space-time region to find the particle is $(\check{t}_0, \check{x}_0)$

$$\begin{aligned}
\check{t}_0 &= t_0 + \alpha \left[\eta^+(\overline{p})^{q-1}\left(\frac{q - \overline{v}}{1 - \overline{v}}\right)\cos\theta - \eta^-(\overline{p}^-)^{n-1}\left(\frac{n + \overline{v}}{1 + \overline{v}}\right)\sin\theta\right], \\
\check{x}_0 &= x_0 + \alpha \left[\eta^+(\overline{p})^{q-1}\left(\frac{q\overline{v} - 1}{1 - \overline{v}}\right)\cos\theta - \eta^-(\overline{p}^-)^{n-1}\left(\frac{n\overline{v} + 1}{1 + \overline{v}}\right)\sin\theta\right].
\end{aligned} \tag{B.11}$$

At constant time, the shift of the wave packet is

$$(\check{x}_0 - x_0) - \overline{v}(\check{t}_0 - t_0) = -\alpha\left[\eta^+(\overline{p})^q\cos\theta + \eta^-(\overline{p}^-)^n\sin\theta\right]/\overline{E}. \tag{B.12}$$

Let's now consider the scattering between two of such wave packets, i and j. Calling the center of scattering (t_{ij}, x_{ij})

$$\begin{pmatrix} t_{ij} \\ x_{ij} \end{pmatrix} = \alpha \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + O(1), \tag{B.13}$$

equal to

$$X_{ij} = \alpha M_{ij}\hat{e}_\theta + O(1) \tag{B.14}$$

$O(1)$ term contain initial conditions which becomes irrelevant since we shall interested in the large α limit. From (B.11) one obtains the coefficients of M_{ij}, that is the matrix which contains the coordinates of the minimum spread

of the wave-packet after the action of Q_θ.
Taking rapidities θ instead of velocities, when $\theta_{i(j)} \to \theta_{i(j)} + k\delta\phi$, for the overlap region too be small enough it must be

$$\parallel \delta M_{ij} \parallel / \parallel M_{ij} \parallel \ll 1. \tag{B.15}$$

Note that δM_{ij} is proportional to $\delta\Phi$ if $|\theta_i - \theta_j| \gg 2k\delta\Phi$ and $1 \gg mk\delta\Phi$, such that if $\delta\Phi$ is small enough, (B.15) holds. When $|\theta_i - \theta_j| \lesssim 2k\delta\Phi$ the two wave packets do not separate at large times.
The matrix M_{ij} has some interesting properties.

1. Given three particles i, j and k such that $M_{ij} = M_{ik}$, if at least $\theta_i \neq \theta_k$ then $M_{ij} = M_{ik} = M_{jk}$.

2. *Isolated-points lemma.* Given three particles i, j and k, if at least if $\theta_i \neq \theta_k$ then $M_{ij} = M_{ik}$ can occur at isolated points in $(\theta_j - \theta_i)$-$(\theta_k - \theta_i)$ space unless particle i has the same $\hat{\eta}^{\pm}$[1] as particle k and $\theta_j = \theta_k$[2].

3. The maximum number of particles that have the same M is 3 times the number of different ratios of $\hat{\eta}^+/\hat{\eta}^-$.

Two-particle scattering.

In general, two ingoing particles scattering results in N outgoing particles

$$\langle \theta_3 \cdots \theta_{N+2} | S | \theta_1 \theta_2 \rangle, \tag{B.16}$$

where $\theta_1 > \theta_2$ and $\theta_3 > \cdots > \theta_{N+2}$. We note that, for *macroscopic* causality properties of the space-time

$$t_{21} \leq t_{23}, \tag{B.17}$$

t_{ij} being the collision time between particles i and j.
As in (B.5) we write

$$\langle \theta_3 \cdots \theta_{N+2} | e^{-i\alpha Q_\theta} S e^{i\alpha Q_\theta} | \theta_1 \theta_2 \rangle, \tag{B.18}$$

According to (B.13) we obtain

$$t_{23} - t_{21} = \alpha \left[(a_{23} - a_{21}) \cos\theta + (b_{23} - b_{21}) \sin\theta \right] + O(1). \tag{B.19}$$

If $a_{23} \neq a_{21}$ and $b_{23} \neq b_{21}$, then, for α large enough $t_{23} - t_{21} < 0$, which is a violation of (B.17). So $a_{23} = a_{21}$ and $b_{23} = b_{21}$, that is $t_{23} = t_{21}$ and

[1] We have defined $\hat{\eta}^+ = \eta^+ \mu^{q-1}$ and $\hat{\eta}^- = \eta^- \mu^{n-1}$.
[2] An isolated point in rapidity differences space is whenever particles have the same rapidities.

$x_{23} = x_{21}\ \forall\theta$. Hence $M_{12} = M_{23}$ and particle 3 comes from the region of collision of particles 1 and 2 and it doesn't interact any more. Since 3 is the fastest particle, this argument holds for every particle, from the fastest to the slowest. Thus, in a two-particle scattering all pairs of particles with different rapidities must have the same M_{ij}. Therefore one or both of the following statement is true:

$$\begin{array}{cccc} (1) & \theta_k \neq \theta_1 & \text{and} & M_{1k} = M_{12}; \\ (2) & \theta_k \neq \theta_2 & \text{and} & M_{2k} = M_{12}; \end{array} \tag{B.20}$$

Because of the isolated-points lemma, also a small variation of $|\theta_1 - \theta_2|$ does not permit any small change of θ_k i order to preserve (B.20), unless

$$\begin{array}{cccc} (1) & \theta_k = \theta_2 & \text{and} & \hat{\eta}_k^{\pm} = \hat{\eta}_2^{\pm}; \\ (2) & \theta_k = \theta_1 & \text{and} & \hat{\eta}_k^{\pm} = \hat{\eta}_1^{\pm}; \end{array} \tag{B.21}$$

An S-matrix can't produce particles in isolated points of the rapidity difference space because obvious continuity and analyticity properties. Therefore no particle production is permitted. Moreover, for the energy-momentum conservation the 2 outgoing particles must have the same momenta of the ingoing particles, thus they must have the same masses. Summarizing, there are two particles in the outgoing state, with the same set of masses, momenta and $\hat{\eta}^{\pm}$

$$\langle p_1 \cdots p_N | S | q_1 q_2 \rangle \propto \delta_{N2} \delta^{(2)}(p_1 - q_1) \delta^{(2)}(p_2 - q_2). \tag{B.22}$$

We mark the non-trivial behavior of the two-particle S-matrix. In fact, if the two in-going particles have equal $\hat{\eta}^{\pm}$, the microscopic time delays or advances can induce interchanges of internal quantum numbers.

Three or more particle scattering

If one demonstrates the factorization for three-particle scattering, then by induction it is possible to turn out the factorization for N-particle.
The coordinate difference of two collisions is

$$X_{ij} - X_{jk} = \alpha \left(M_{ij} - M_{jk} \right) \hat{e}_\theta + O(1). \tag{B.23}$$

By choosing $\hat{e}_\theta$ to be different than the only one null eigenvector, (B.23) can be made arbitrarily far apart by acting on α. In doing so, it is easy to see that the 3-particle scattering can be separated into two 2-particle consequent scattering, for which factorization holds. If $M_{ij} = M_{jk}$, from the isolated-points lemma two ingoing particles have the same rapidity and can be considered like two separate ingoing states.

Summarizing, the S-matrix of a $1+1$ dimensional theory which has at least two conserved charges different from Energy and Momentum is factorizable in 2-particle S-matrix and the scattering doesn't permit particle creation.

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